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On two-parameter global bifurcation of periodic solutions to a class of differential variational inequalities

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ABSTRACT

In this paper, by using the method of integral guiding functions a multiparameter global bifurcation theorem for differential inclusions with the periodic condition is proved. It is shown how the abstract result can be applied to the study of the two-parameter global bifurcation of periodic solutions for a class of differential variational inequalities.

1. Introduction

Let $K \subset \mathbb{R}^m$ be a closed, convex cone and $\langle \cdot, \cdot \rangle = [| \cdot |]$ denoted the inner product [respectively, norm] in finite-dimensional spaces. In this paper, we study a global bifurcation of periodic solutions of differential variational inequalities (DVIs) of the form:

$$\begin{cases} x'(t) = A(\mu)x(t) + f\big(t, x(t), u(t), \mu\big) & \text{for a.e. } t \in [0,T], \\ \Big\langle \widetilde{u} - u(t), G\big(t, x(t), \mu\big) + F(u(t)) \Big\rangle \geq 0 & \text{for a.e. } t \in [0,T], \ \forall\, \widetilde{u} \in K, \\ x(0) = x(T) \quad \text{and} \quad u(t) \in K, \end{cases} \tag{1.1}$$

where $f\colon [0,T] \times \mathbb{R}^2 \times \mathbb{R}^m \times \mathbb{R}^2 \to \mathbb{R}^2, G\colon [0,T] \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^m$, and $F\colon \mathbb{R}^m \to \mathbb{R}^m$ are *given continuous maps*; $A\colon \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ is a map defined as

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$$A(\mu) = \begin{pmatrix} c\mu_1 & c\mu_2 \\ c\mu_2 & -c\mu_1 \end{pmatrix}, \quad \mu = (\mu_1, \mu_2), \tag{1.2}$$

while $c > 0$ is a fixed number and $\mu \in \mathbb{R}^2$ is a parameter.

The notion of DVIs was used by Aubin and Cellina [3] in 1984. However, DVIs were first systematically studied by Pang and Stewart [33]. It is shown that DVIs are useful for representing many applied mathematical models including the differential complementarity problem, the variational inequality of evolution and the extended system (see [33] for more detail).

Since K is a closed convex cone the variational inequality problem and the complementarity problem have the same solution set (see [29]). Hence, problem (1.1) is equivalent to the following nonlinear *differential complementarity problem*

$$\begin{cases} x'(t) = A(\mu)x(t) + f\big(t, x(t), u(t), \mu\big) & \text{for a.e. } t \in [0,T], \\ K \ni u(t) \perp G\big(t, x(t), \mu\big) + F(u(t)) \in K^* & \text{for a.e. } t \in [0,T], \\ x(0) = x(T), \end{cases} \tag{1.3}$$

where K^* is the dual cone of K.

The early version of the differential complementarity problem is the *variational inequality of evolution* which was introduced by Henry [24,25] as a class of differential inclusions known as projected differential inclusions. Among a large number of works concerning the differential complementarity problem let us mention the work of Hipfel [26] for the nonlinear differential complementarity problem, the work of Heemels [23] and Camlibel [7] for the linear complementarity problem. The stability theory for differential complementarity problem is investigated in [1,8,19,17,18] (see also the references therein).

For the study of the global bifurcation of solutions of problem (1.1) (or equivalently, problem (1.3)) we replace (1.1) by a parameterized family of differential inclusions, and then the global bifurcation theory for inclusions is applied. Let us recall that the multiparameter global bifurcation problem for inclusions with compact convex set-valued mappings was first studied by Alexander and Fitzpatrick [2]. Górniewicz and Kryszewski [21,22] extended the result for the case of acyclic set-valued mappings in finite-dimensional spaces. Recently, Gabor and Kryszewski [14,15] studied the multiparameter global bifurcation problem for inclusions with linear Fredholm maps of nonnegative index.

One frequently occurred in the study of global bifurcation is the problem of evaluation of that called *global bifurcation index* (see, e.g. [14,30,32]). In [30] Kryszewski used the method of guiding functions to evaluate the global bifurcation index and to describe the global structure of branches of periodic orbits for families of differential inclusions of the form

$$\begin{cases} u'(t) \in g(t, u(t), \lambda), \\ u(0) = u(T), \end{cases} \tag{1.4}$$

where $g\colon [0,T] \times \mathbb{R}^n \times \mathbb{R}^k \multimap \mathbb{R}^n$ is a upper-Carathéodory multivalued mapping with compact and convex values. Another construction for the evaluation of the global bifurcation index (for one-parameter case of (1.4)) via guiding functions and integral guiding functions was suggested by Loi and Obukhovskii (see [32]). This construction also applied to study the global bifurcation of periodic solutions of DVIs with one-parameter (see [31]).

In the present paper, after necessary preliminaries, in Section 3 by using the method of integral guiding functions we evaluate the global bifurcation index at $(0,0)$ for problem (1.4) via the index of the guiding function. From the fact that the index of the guiding function is a non-zero element we describe the global bifurcation of solutions of (1.4). In Section 4 it is shown how the abstract result can be applied to study the global bifurcation of solutions of (1.1).

2. Preliminaries

2.1. Notation

Throughout this paper by the symbol $\mathcal{C}$ we denote the space $C([0,T];\mathbb{R}^n)$ of continuous functions and by $\mathcal{L}^p$ $(p \geq 1)$ the space $L^p([0,T];\mathbb{R}^n)$ of pth integrable functions with usual norms:

$$\|x\|_{\mathcal{C}} = \max_{t\in[0,T]} |x(t)| \quad \text{and} \quad \|f\|_p = \left(\int_0^T |f(s)|^p \, ds\right)^{\frac{1}{p}}.$$

An *open ball* of radius r centered at 0 in $\mathcal{C}$ $[\mathbb{R}^n]$ is denoted by $B_{\mathcal{C}}(0,r)$ [respectively, $B^n(0,r)$]. The unit open ball [unit sphere] in $\mathbb{R}^n$ are denoted by B^n [resp., S^{n-1}].

Consider the space of all absolutely continuous functions $x : [0,T] \to \mathbb{R}^n$ whose derivatives belong to $\mathcal{L}^p$. It is known (see, e.g. [4]) that this space can be identified with the Sobolev space $W^{1,p}([0,T];\mathbb{R}^n)$ endowed with the norm

$$\|x\|_{\mathcal{W}} = \left(\|x\|_p^p + \|x'\|_p^p\right)^{\frac{1}{p}}.$$

We will denote this space by $\mathcal{W}^{1,p}$. Notice that (see, e.g. [10]), for the case $p = 2$, the embedding $\mathcal{W}^{1,2} \hookrightarrow \mathcal{C}$ is compact. By the symbol $\mathcal{W}_T^{1,p}$ we will denote the subspace of all functions $x \in \mathcal{W}^{1,p}$ such that $x(0) = x(T)$.

2.2. Multimaps

Let X, Y be Banach spaces. Denote by $P(Y)$ $[Cv(Y), Kv(Y)]$ the collections of all nonempty [respectively, nonempty closed convex, nonempty compact convex] subsets of Y.

Definition 1 (*See, e.g. [5,6,20,28]*)**.** A multivalued mapping (multimap) $\Sigma\colon X \to P(Y)$ is said to be: (i) upper semicontinuous (u.s.c.) if for every open subset $V \subset Y$ the set

$$\Sigma_+^{-1}(V) = \{x \in X\colon \Sigma(x) \subset V\}$$

is open in X; (ii) closed if its graph $\{(x,y) \in X \times Y : y \in \Sigma(x)\}$ is a closed subset of $X \times Y$; (iii) compact, if the set $\Sigma(X)$ is relatively compact in Y.

Definition 2. Let $\mathcal{F} : X \to P(Y)$ be a multimap. For a given $\varepsilon > 0$, a continuous map $f\colon X \to Y$ is called an ε-approximation of the multimap $\mathcal{F}$ if for each $x \in X$ there exists $x' \in X$ such that $\varrho_X(x,x') < \varepsilon$ and

$$f(x) \in O_\varepsilon(\mathcal{F}(x')),$$

for all $x \in X$, where $O_\varepsilon(M)$ is the ε-neighborhood of the set M.

It is easy to see that the ε-approximation may be equivalently defined as the map whose graph belongs to the ε-neighborhood of the graph of the corresponding multimap.

Proposition 1 (*See, e.g. [6,28]*)**.** *For each u.s.c. multimap $\mathcal{F}\colon X \to Cv(Y)$ and $\varepsilon > 0$ there exists a continuous map $f_\varepsilon\colon X \to Y$ such that*

(i) *for every $x \in X$ there exists $x' \in X$ such that $\rho(x,x') < \varepsilon$ and*

$$f_\varepsilon(x) \cup \mathcal{F}(x) \subset O_\varepsilon(\mathcal{F}(x'));$$

(ii) *$f_\varepsilon(X) \subset co\mathcal{F}(X)$, where co denotes the convex hull of a set.*

2.3. Variational inequalities

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$; $K \subset H$ be a closed convex subset. Let us recall that a mapping $g\colon K \to H$ is said to be:

(a) *continuous on finite-dimensional subspaces* if it is continuous on $K \cap U$ for every finite-dimensional subspace U of H with $K \cap U \neq \emptyset$. (Notice that every continuous mapping from K to H is continuous on finite-dimensional subspaces)

(b) *pseudo-monotone* if for any $x, y \in K$,

$$\big\langle x - y, g(y) \big\rangle \geq 0 \quad \text{implies } \big\langle x - y, g(x) \big\rangle \geq 0.$$

(c) *monotone* if

$$\big\langle x - y, g(x) - g(y) \big\rangle \geq 0 \quad \text{for all } x, y \in K.$$

(d) *strongly monotone* if there exists $\beta > 0$ such that

$$\big\langle x - y, g(x) - g(y) \big\rangle \geq \beta \|x - y\|^2 \quad \text{for all } x, y \in K.$$

(e) *coercive* if

$$\frac{\big\langle x, g(x) \big\rangle}{\|x\|} \to \infty \quad \text{as } \|x\| \to \infty \text{ and } x \in K.$$

Lemma 1 (*See, Lemma 2.1 [9]*). *Let $g\colon K \to H$ be a pseudo-monotone mapping which is continuous on finite-dimensional subspaces. Then $w \in K$ is a solution of*

$$\big\langle u - w, g(w) \big\rangle \geq 0 \quad \textit{for all } u \in K$$

if and only if

$$\big\langle u - w, g(u) \big\rangle \geq 0 \quad \textit{for all } u \in K.$$

Lemma 2 (*See, Lemma 3.1 [29]*). *Let K be a closed convex cone in H and g a mapping from K to H. Then $w^* \in K$ satisfies*

$$\big\langle u - w^*, g(w^*) \big\rangle \geq 0 \quad \textit{for all } u \in K$$

if and only if

$$g(w^*) \in K^* \quad \textit{and} \quad \big\langle w^*, g(w^*) \big\rangle = 0,$$

where K^* is the dual cone of K.

Lemma 3 (*See, Theorem 3.2 [9]*). *Let K be a closed convex cone in H and $g\colon K \to H$ be a coercive and pseudo-monotone mapping which is continuous on finite-dimensional subspaces. Then there exists $x^* \in K$ such that*

$$\big\langle u - x^*, g(x^*) \big\rangle \geq 0 \quad \textit{for all } u \in K.$$

2.4. Linear Fredholm operators

Let X, Y be Banach spaces.

Definition 3 (*See, e.g. [16]*)**.** A linear bounded operator $L\colon domL \subseteq X \to Y$ is called Fredholm of index q ($q \geq 0$) if

(1i) ImL is closed in Y;
(2i) $KerL$ and $CokerL = Y/ImL$ have finite dimensions and, moreover,

$$dim\, KerL - dim\, CokerL = q$$

Let $L\colon domL \subseteq X \to Y$ be a linear Fredholm operator of index q, then there exist projectors $P_L\colon X \to X$ and $Q_L\colon Y \to Y$ such that $ImP_L = KerL$ and $KerQ_L = ImL$. If the operator

$$L_{P_L}\colon domL \cap KerP_L \to ImL$$

is defined as the restriction of L on $domL \cap KerP_L$ then it is clear that L_{P_L} is an algebraic isomorphism and we may define $K_{P_L}\colon ImL \to domL$ as $K_{P_L} = L_{P_L}^{-1}$.

For the case $q = 0$, if we let $\Pi_L\colon Y \to CokerL$ be the canonical surjection:

$$\Pi_L z = z + ImL$$

and $\Lambda_L\colon CokerL \to KerL$ be a one-to-one linear mapping, then the equation

$$Lx = y, \quad y \in Y$$

is equivalent to the equation

$$(i - P_L)x = (\Lambda_L \Pi_L + K_L)y,$$

where i denotes the identity operator and $K_L\colon Y \to X$ be defined as

$$K_L = K_{P_L}(i - Q_L).$$

2.5. Coincidence index

For reader's convenience, we recall in this section the definition of coincidence index presented in [30] (see also [11–13]). Firstly, let us recall the definition of the topological degree of a continuous map between two finite-dimensional spaces (for the notion of the homotopy groups and the cohomotopy sets we refer readers to [27,34]).

Let $U \subset \mathbb{R}^m$ be an open bounded subset and $f\colon \overline{U} \to \mathbb{R}^n$ a continuous map, where $m \geq n \geq 1$. Assume that $f(x) \neq 0$ for all x belonging to the boundary ∂U of the set U. Therefore, the distance $d(0, f(\partial U))$ from 0 to the set $f(\partial U)$ in $\mathbb{R}^n$ is positive. Taking $\rho = \frac{1}{2} d(0, f(\partial U))$, we obtain $f(\partial U) \subset \mathbb{R}^n \setminus B^n(0,\rho)$. Hence, the map

$$f\colon (\overline{U}, \partial U) \longrightarrow (\mathbb{R}^n, \mathbb{R}^n \setminus B^n(0,\rho))$$

induces a map between cohomotopy sets

$$f^\sharp\colon \pi^n(\mathbb{R}^n, \mathbb{R}^n \setminus B^n(0,\rho)) \longrightarrow \pi^n(\overline{U}, \partial U).$$

Consider the following sequence of maps

$$\pi^n(\mathbb{R}^n, \mathbb{R}^n \setminus B^n(0,\rho)) \xrightarrow{f^\sharp} \pi^n(\overline{U}, \partial U) \xleftarrow{i_1^\sharp}$$

$$\xleftarrow{i_1^\sharp} \pi^n\big(\mathbb{R}^m, \mathbb{R}^m \setminus U\big) \xrightarrow{i_2^\sharp} \pi^n\big(\mathbb{R}^m, \mathbb{R}^m \setminus B^m(0,r)\big),$$

where $r > 0$ is such that $U \subset B^m(0,r)$,

$$i_1 \colon \big(\overline{U}, \partial U\big) \longrightarrow \big(\mathbb{R}^m, \mathbb{R}^m \setminus U\big),$$

and

$$i_2 \colon \big(\mathbb{R}^m, \mathbb{R}^m \setminus B^m(0,r)\big) \longrightarrow \big(\mathbb{R}^m, \mathbb{R}^m \setminus U\big)$$

are inclusion maps.

The map $i_1^\sharp$ is a bijection (by the excision property), therefore according to the relations $\pi^n(S^n) = \pi^n\big(\mathbb{R}^n, \mathbb{R}^n \setminus B^n(0,\rho)\big)$ and $\pi^n(S^m) = \pi^n\big(\mathbb{R}^m, \mathbb{R}^m \setminus B^m(0,r)\big)$, the map

$$\omega_f = i_2^\sharp \circ (i_1^\sharp)^{-1} \circ f^\sharp \colon \pi^n(S^n) \longrightarrow \pi^n(S^m)$$

is well-defined.

Definition 4. The element $\omega_f(\mathbf{1}) \in \pi^n(S^m) = \pi_m(S^n)$ is called the *topological degree* of the map f on $\overline{U}$ and it is denoted by $deg\,(f, \overline{U})$, where $\mathbf{1}$ is the homotopy class of the identity map $id \colon S^n \to S^n$ in $\pi^n(S^n) \cong \mathbb{Z}$.

Notice that the topological degree $deg(f, \overline{U})$ does not depend on the choice of $r > 0$. For illustration of the above definition let us recall an example presented in [14, Example 4.1]

Example 1. Let $U = B^m$ and $\overline{f} = f_{|_{S^{m-1}}} \colon S^{m-1} \to \mathbb{R}^n \setminus B^n(0,\rho)$, where ρ is taken as above. Consider the diagram

$$\begin{array}{ccccc}
\pi^n(S^n) \cong \pi^n(\mathbb{R}^n, \mathbb{R}^n \setminus B^n(0,\rho)) & \xrightarrow{f^\sharp} & \pi^n(\overline{B^m}, S^{m-1}) \cong \pi^n(S^m) & \xrightarrow{=} & \pi_m(S^n) \\
\cong \Big\uparrow \delta_1 & & \Big\uparrow \delta & & \Big\uparrow \Sigma \\
\pi^{n-1}(S^{n-1}) \cong \pi^{n-1}(\mathbb{R}^n \setminus B^n(0,\rho)) & \xrightarrow{\overline{f}^\sharp} & \pi^{n-1}(S^{m-1}) & \xrightarrow{=} & \pi_{m-1}(S^{n-1})
\end{array}$$

where δ, δ_1 are the respective coboundary operators and Σ is the suspension homomorphism. This diagram is commutative. Therefore, if $f \colon (\overline{B^m}, S^{m-1}) \to (\mathbb{R}^n, S^{n-1})$, then $deg(f, \overline{B^m}) = \Sigma([\overline{f}]) \in \pi_m(S^n)$, where $[\overline{f}] \in \pi_{m-1}(S^{n-1})$ is the homotopy class of $\overline{f} \colon S^{m-1} \to S^{n-1}$.

Now let X, Y be Banach spaces; $U \subset X$ an open bounded subset; $L \colon X \to Y$ a linear Fredholm map of index $q \geq 0$ and $F \colon \overline{U} \to Kv(Y)$ a compact u.s.c. multimap such that $Lx \notin F(x)$ for all $x \in \partial U$. Let $\delta = \frac{1}{2} dist_Y\big(0, (L-F)(\partial U)\big)$. For $\varepsilon \in (0, \delta]$ let $p_\varepsilon \colon \overline{F(U)} \to Y$ be the Schauder projection of the compact set $\overline{F(U)}$ into a finite-dimensional subspace Z of Y such that $\|p_\varepsilon y - y\| < \varepsilon$ for all $y \in \overline{F(U)}$. Denote by W' the finite-dimensional subspace of $Im\,L$ such that $Z \subset W = W' \oplus Im(Q_L)$. Set $T = L^{-1}(W), U_T = U \cap T$. It is clear that $L_{|_T} \colon T \to W$ is Fredholm operator of index q and

$$dim\,T = dim\,W + q.$$

W.l.o.g. assume that $dim\,W = n \geq q + 2$. Then the coincidence index $Ind(L, F, \overline{U})$ is defined as

Definition 5.

$$Ind(L, F, \overline{U}) := deg(L - p_\varepsilon \circ f_\kappa, \overline{U_T}) \in \pi^n(S^{n+q}) \cong \Pi_q,$$

where f_κ is an κ-approximation of F on $\overline{U_T}$ while $\kappa \in (0, \varepsilon)$ is sufficiently small and Π_q denotes qth stable homotopy group of spheres (see, e.g. [27]).

The given coincidence index has the following properties.

(i) (Existence) If $Ind(L, F, \overline{U}) \neq 0 \in \Pi_q$, then there exists $x \in U$ such that $Lx \in F(x)$.
(ii) (Localization) If $U' \subset U$ is open and

$$C := \{x \in U : Lx \in F(x)\} \subset U',$$

then $Ind(L, F, \overline{U}) = Ind(L, F, \overline{U'})$.
(ii) (Additivity) If U_1, U_2 are open bounded disjoint subsets of X and $U = U_1 \cup U_2$, then

$$Ind(L, F, \overline{U}) = Ind(L, F, \overline{U_1}) + Ind(L, F, \overline{U_2}).$$

(iii) (Restriction) If $F(\overline{U})$ belongs to a subspace Y' of Y, then

$$Ind(L, F, \overline{U}) = Ind(L, F, \overline{U_T}),$$

where $U_T = U \cap T, T = L^{-1}(Y')$.
(iii) (Homotopy) If there exists a compact u.s.c. multimap $\Phi : \overline{U} \times [0, 1] \to Kv(Y)$ such that $Lx \notin \Phi(x, \lambda)$ for all $(x, \lambda) \in \partial U \times [0, 1]$, then

$$Ind\big(L, \Phi(\cdot, 0), \overline{U}\big) = Ind\big(L, \Phi(\cdot, 1), \overline{U}\big).$$

3. A multiparameter global bifurcation theorem

Consider a family of differential inclusions of the form

$$\begin{cases} x'(t) \in \mathcal{F}(t, x(t), \mu) & \text{for a. e. } t \in [0, T], \\ x(0) = x(T), \end{cases} \tag{3.1}$$

where $\mathcal{F} : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \to Kv(\mathbb{R}^n)$ be such that

($\mathcal{F}1$) for every $(z, \mu) \in \mathbb{R}^n \times \mathbb{R}^k$ multifunction $\mathcal{F}(\cdot, z, \mu) : [0, T] \to Kv(\mathbb{R}^n)$ has a measurable selection;
($\mathcal{F}2$) for a.e. $t \in [0, T]$ multimap $\mathcal{F}(t, \cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^k \to Kv(\mathbb{R}^n)$ is u.s.c.;
($\mathcal{F}3$) for every bounded subset $\Omega \subset \mathbb{R}^n \times \mathbb{R}^k$ there exists a positive function $\gamma_\Omega \in L^2[0, T]$ such that

$$\|\mathcal{F}(t, z, \mu)\| := \max\{|y| : y \in \mathcal{F}(t, z, \mu)\} \leq \gamma_\Omega(t),$$

for all $(z, \mu) \in \Omega$ and a.e. $t \in [0, T]$;
($\mathcal{F}4$) $0 \in \mathcal{F}(t, 0, \mu)$ for all $\mu \in \mathbb{R}^k$ and a.e. $t \in [0, T]$.

From conditions ($\mathcal{F}1$)–($\mathcal{F}3$) it follows that the superposition multioperator

$$\begin{aligned} &\mathcal{P}_\mathcal{F} : \mathcal{C} \times \mathbb{R}^k \to Cv(\mathcal{L}^2), \\ &\mathcal{P}_\mathcal{F}(x, \mu) = \big\{f \in \mathcal{L}^2 : f(t) \in \mathcal{F}(t, x(t), \mu) \text{ for a.e. } t \in [0, T]\big\}, \end{aligned}$$

is defined and closed (see, e.g. [5,6,20,28]).

Define the map $L : \mathcal{W}_T^{1,2} \to \mathcal{L}^2, Lx = x'$. It is clear that L is a linear Fredholm operator of index zero and

$$Ker\, L \cong \mathbb{R}^n \cong Coker\, L.$$

The projection

$$\Pi_L : \mathcal{L}^2 \to \mathbb{R}^n,$$

is defined as

$$\Pi_L(f) = \frac{1}{T}\int_0^T f(s)\,ds$$

and the homeomorphism $\Lambda_L : \mathbb{R}^n \to \mathbb{R}^n$ is an identity operator.

Problem (3.1) can be substituted by the following problem

$$Lx \in \mathcal{P}_{\mathcal{F}}(x,\mu),$$

or equivalently,

$$x \in G(x,\mu), \tag{3.2}$$

where

$$G : \mathcal{C} \times \mathbb{R}^k \to Kv(\mathcal{C}),$$
$$G(x,\mu) = P_L x + (\Pi_L + K_L) \circ \mathcal{P}_{\mathcal{F}}(x,\mu).$$

Definition 6. By a solution to problem (3.1) we mean a pair $(x,\mu) \in \mathcal{C} \times \mathbb{R}^k$ that satisfies (3.2).

It is clear that problem (3.1) has the trivial solution $(0,\mu)$ for all $\mu \in \mathbb{R}^k$. Denote by $\mathcal{S}$ the set of all nontrivial solutions of (3.1).

Definition 7. A point $(0,\mu_0) \in \mathcal{C} \times \mathbb{R}^k$ is said to be a bifurcation point of problem (3.1) if for every open bounded subset $U \subset \mathcal{C} \times \mathbb{R}^k$ containing $(0,\mu_0)$, there exists a solution $(x,\mu) \in U$ to problem (3.1) such that $x \neq 0$.

Definition 8. A family of continuously differentiable functions $V_\mu : \mathbb{R}^n \to \mathbb{R}, \mu \in \mathbb{R}^k$, is said to be a family of local integral guiding functions for (3.1) at $(0,0)$, if there exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0,\varepsilon_0)$ there is a sufficiently small number $\delta_\varepsilon > 0$ (which continuously depends on ε) such that for every $x \in \mathcal{C}, 0 < \|x\|_{\mathcal{C}} \leq \delta_\varepsilon$, the following relation holds

$$\int_0^T \big\langle \nabla V_\mu(x(s)), f(s) \big\rangle\,ds > 0$$

for all $\mu \in S^{k-1}(0,\varepsilon)$ and $f \in \mathcal{P}_{\mathcal{F}}(x,\mu)$, where ∇V_μ denotes the gradient of V_μ.

Lemma 4. *If V_μ is a family of local integral guiding functions for (3.1) at $(0,0)$, then for every $\varepsilon \in (0,\varepsilon_0)$:*

(a) *inclusion (3.1) has only trivial solutions $(0,\mu)$ on $\overline{B_{\mathcal{C}}(0,\delta_\varepsilon)} \times S^{k-1}(0,\varepsilon)$;*
(b) *equation $\nabla V_\mu(w) = 0$ has only trivial solutions on $\overline{B^n(0,\delta_\varepsilon)}$ for all $\mu \in S^{k-1}(0,\varepsilon)$.*

Proof. (a) Assume that $(x,\mu) \in \overline{B_{\mathcal{C}}(0,\delta_\varepsilon)} \times \mathbb{R}^k$, $|\mu| = \varepsilon$, is a nontrivial solution to (3.1). Therefore, there exists $f \in \mathcal{P}_{\mathcal{F}}(x,\mu)$ such that $x'(t) = f(t)$ for a.e. $t \in [0,T]$. Since $|\mu| = \varepsilon$ and $0 < \|x\|_{\mathcal{C}} \leq \delta_\varepsilon$ we have

$$\int_0^T \big\langle \nabla V_\mu(x(t)), f(t) \big\rangle dt > 0.$$

On the other hand,

$$\int_0^T \langle \nabla V_\mu(x(t)), f(t)\rangle dt = \int_0^T \langle \nabla V_\mu(x(t)), x'(t)\rangle dt = V_\mu(x(T)) - V_\mu(x(0)) = 0,$$

giving a contradiction.

(b) Assume that $w \in \overline{B^n(0, \delta_\varepsilon)}$ is a nontrivial solution to equation $\nabla V_\mu(w) = 0$ for some $\mu \in S^{k-1}(0, \varepsilon)$. For any $f \in \mathcal{P}_{\mathcal{F}}(w, \mu)$, since $|\mu| = \varepsilon$ and $0 < \|w\|_{\mathcal{C}} = |w| \le \delta_\varepsilon$ we have

$$\int_0^T \langle \nabla V_\mu(w), f(t)\rangle dt > 0.$$

Therefore $\nabla V_\mu(w) \neq 0$, that is a contradiction. ■

For each $\varepsilon \in (0, \varepsilon_0)$, set $O_\varepsilon = B^{n+k}\big(0, \sqrt{\varepsilon^2 + \delta_\varepsilon^2}\,\big)$ and define the map

$$\widetilde{V}_\varepsilon \colon \overline{O_\varepsilon} \to \mathbb{R}^{n+1},$$
$$\widetilde{V}_\varepsilon(w, \mu) = \{-\nabla V_\mu(w)\} \times \{\varepsilon^2 - |\mu|^2\}.$$

From Lemma 4 it follows that $\widetilde{V}_\varepsilon$ has no zeros on the sphere ∂O_ε. Hence, the topological degree

$$deg(\widetilde{V}_\varepsilon, \overline{O_\varepsilon}) = \omega_{\widetilde{V}_\varepsilon}(\mathbf{1}) \in \pi^{n+1}(S^{n+k}) = \pi_{n+k}(S^{n+1})$$

is well-defined.

Let us show that this degree does not depend on the choice of $\varepsilon \in (0, \varepsilon_0)$. In fact, let $\varepsilon_1, \varepsilon_2, 0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_0$, be arbitrary numbers. For each $\lambda \in [0, 1]$, set $\varepsilon_{\lambda+1} = \lambda\varepsilon_2 + (1-\lambda)\varepsilon_1$,

$$O_{\varepsilon_{\lambda+1}} = B^{n+k}\Big(0, \sqrt{\varepsilon_{\lambda+1}^2 + \delta_{\varepsilon_{\lambda+1}}^2}\,\Big),$$

where $\delta_{\varepsilon_{\lambda+1}}$ is the constant from Definition 8, and consider the map

$$V_\lambda^\sharp \colon \overline{O_{\varepsilon_{\lambda+1}}} \to \mathbb{R}^{n+1},$$
$$V_\lambda^\sharp(w, \mu) = \{-\nabla V_\mu(w)\} \times \{\varepsilon_{\lambda+1}^2 - |\mu|^2\}.$$

Assume that there exist $\lambda_* \in [0, 1]$ and $(w_*, \mu_*) \in \partial O_{\varepsilon_{\lambda_*+1}}$ such that

$$V_{\lambda_*}^\sharp(w_*, \mu_*) = 0,$$

or equivalently,

$$\begin{cases} \nabla V_{\mu_*}(w_*) = 0, \\ |\mu_*| = \varepsilon_{\lambda_*+1}. \end{cases}$$

From $(w_*, \mu_*) \in \partial O_{\varepsilon_{\lambda_*+1}}$ it follows that $|w_*| = \delta_{\varepsilon_{\lambda_*+1}}$. That contradicts to Lemma 4(b). Hence, the topological degree $deg(\widetilde{V}_\varepsilon, \overline{O_\varepsilon})$ is the same for all $\varepsilon \in (0, \varepsilon_0)$. This element is called *the index of the family of local integral guiding functions* V_μ and is denoted by $ind\, V_\mu$.

Theorem 1. *Let conditions* $(\mathcal{F}1)$–$(\mathcal{F}4)$ *hold. In addition, assume that there exists a family of local integral guiding functions* V_μ *for* (3.1) *at* $(0, 0)$ *such that* $ind\, V_\mu \neq 0$. *Then there is a connected subset* $\mathcal{R} \subset \mathcal{S}$ *such that* $(0, 0) \in \overline{\mathcal{R}}$ *and either* $\mathcal{R}$ *is unbounded or* $\overline{\mathcal{R}} \ni (0, \mu_*)$ *for some* $\mu_* \neq 0$.

Proof. Let us represent the space $\mathcal{L}^2$ as

$$\mathcal{L}^2 = \mathcal{L}_0 \oplus \mathcal{L}_1,$$

where $\mathcal{L}_0 = Coker\, L$ and $\mathcal{L}_1 = Im\, L$. The decomposition of an element $f \in \mathcal{L}^2$ is denoted by

$$f = f_0 + f_1, \quad f_0 \in \mathcal{L}_0, f_1 \in \mathcal{L}_1.$$

Define the map

$$\ell\colon \mathcal{C} \times \mathbb{R}^k \to \mathcal{C} \times \mathbb{R}, \quad \ell(x,\mu) = (x, 0).$$

For $r, \varepsilon > 0$ set

$$B_{r,\varepsilon} = \big\{(x,\mu) \in \mathcal{C} \times \mathbb{R}^k\colon \|x\|^2_{\mathcal{C}} + |\mu|^2 \le r^2 + \varepsilon^2\big\},$$

and consider the multimap

$$G_r\colon B_{r,\varepsilon} \to Kv(\mathcal{C} \times \mathbb{R}),$$
$$G_r(x,\mu) = \{G(x,\mu)\} \times \{r^2 - \|x\|^2_{\mathcal{C}}\}.$$

It is obvious that ℓ is a linear Fredholm map of index $k-1$.

Step 1. We will show that G_r is a compact u.s.c. multimap. Indeed, since the superposition multioperator $\mathcal{P}_{\mathcal{F}}$ is closed and the operator $\Pi_L + K_L$ is linear and continuous, the multimap $(\Pi_L + K_L) \circ \mathcal{P}_{\mathcal{F}}$ is closed (see, e.g. Theorem 1.5.30 [6]). Now, by virtue of $(\mathcal{F}3)$ the set $\mathcal{P}_{\mathcal{F}}(B_{r,\varepsilon})$ is bounded in $\mathcal{L}^2$. Therefore, the set $(\Pi_L + K_L) \circ \mathcal{P}_{\mathcal{F}}(B_{r,\varepsilon})$ is bounded in $\mathcal{W}^{1,2}_T$. From the compactness of the embedding $\mathcal{W}^{1,2} \hookrightarrow \mathcal{C}$ and the fact that the map P_L has its range in a finite-dimensional space it follows that the set $G(B_{r,\varepsilon})$ is relatively compact in $\mathcal{C}$. Compact and closed multimap is u.s.c. So, the restriction $G_{|B_{r,\varepsilon}}$, and hence G_r, is a compact u.s.c. multimap.

Step 2. Choosing arbitrarily $\varepsilon \in (0, \varepsilon_0)$ and sufficiently small $r \in (0, \delta_\varepsilon)$, where ε_0 and δ_ε are the constants from Definition 8, we will show that $\ell(x,\mu) \notin G_r(x,\mu)$ for all $(x,\mu) \in \partial B_{r,\varepsilon}$.

Indeed, assume to the contrary that there is $(x,\mu) \in \partial B_{r,\varepsilon}$ such that $\ell(x,\mu) \in G_r(x,\mu)$. Then,

$$x \in G(x,\mu), \tag{3.3}$$

and

$$\|x\|_{\mathcal{C}} = r. \tag{3.4}$$

From (3.3) it follows that there is $f \in \mathcal{P}_{\mathcal{F}}(x,\mu)$ such that $x'(t) = f(t)$ for a.e. $t \in [0,T]$.

Applying (3.4), we obtain $|\mu| = \varepsilon$. Moreover, from $\|x\|_{\mathcal{C}} = r < \delta_\varepsilon$ we have

$$\int_0^T \big\langle \nabla V_\mu(x(s)), f(s)\big\rangle ds > 0$$

for all $\mu \in S^{k-1}(0,\varepsilon)$.

On the other hand,

$$\int_0^T \big\langle \nabla V_\mu(x(s)), f(s)\big\rangle ds = \int_0^T \big\langle \nabla V_\mu(x(s)), x'(s)\big\rangle ds = 0,$$

giving a contradiction. Therefore, the coincidence index $Ind(\ell, G_r, B_{r,\varepsilon})$ is well-defined for each $\varepsilon \in (0,\varepsilon_0)$ and $r \in (0,\delta_\varepsilon)$.

Step 3. Fix $\varepsilon \in (0, \varepsilon_0)$ and $r \in (0, \delta_\varepsilon)$. Let us evaluate $Ind(\ell, G_r, B_{r,\varepsilon})$. Toward this goal, consider the multimap $\Sigma\colon B_{r,\varepsilon} \times [0,1] \to Kv(\mathcal{C} \times \mathbb{R})$,

$$\Sigma(x, \mu, \lambda) = \big\{P_L x + (\Pi_L + K_L) \circ \alpha\big(\mathcal{P}_{\mathcal{F}}(x, \mu), \lambda\big)\big\} \times \{\tau\},$$

where

$$\tau = \lambda(r^2 - \|x\|^2_{\mathcal{C}}) + (1 - \lambda)(|\mu|^2 - \varepsilon^2),$$

and $\alpha\colon \mathcal{L}^2 \times [0,1] \to \mathcal{L}^2$,

$$\alpha(f, \lambda) = f_0 + \lambda f_1, \quad f_0 \in \mathcal{L}_0, f_1 \in \mathcal{L}_1, f = f_0 + f_1.$$

Following Step 1 we can easily prove that Σ is a compact u.s.c. multimap.

Assume $(x^*, \mu^*, \lambda_*) \in \partial B_{r,\varepsilon} \times [0,1]$ is such that $\ell(x^*, \mu^*) \in \Sigma(x^*, \mu^*, \lambda^*)$. Then

$$\lambda^*(r^2 - \|x^*\|^2_{\mathcal{C}}) + (1 - \lambda^*)(|\mu^*|^2 - \varepsilon^2) = 0, \tag{3.5}$$

and there is a function $f^* \in \mathcal{P}_{\mathcal{F}}(x^*, \mu^*)$ such that

$$x^* = P_L x^* + (\Pi_L + K_L) \circ \alpha(f^*, \lambda^*)$$

or equivalently,

$$\begin{cases} (x^*)' = \lambda^* f_1^* \\ 0 = f_0^*, \end{cases}$$

where $f_0^* + f_1^* = f^*, f_0^* \in \mathcal{L}_0$ and $f_1^* \in \mathcal{L}_1$.

From $(x^*, \mu^*) \in \partial B_{r,\varepsilon}$ it follows that

$$r^2 - \|x^*\|^2_{\mathcal{C}} = |\mu^*|^2 - \varepsilon^2.$$

Hence, from (3.5) we obtain

$$\|x^*\|_{\mathcal{C}} = r \quad \text{and} \quad |\mu^*| = \varepsilon.$$

From the choice of r it follows that

$$\int_0^T \big\langle \nabla V_{\mu^*}(x^*(s)), f(s)\big\rangle ds > 0 \quad \text{for all } f \in \mathcal{P}_{\mathcal{F}}(x^*, \mu^*).$$

If $\lambda^* \neq 0$: then

$$\begin{aligned} \int_0^T \big\langle \nabla V_{\mu^*}(x^*(s)), f^*(s)\big\rangle\, ds &= \int_0^T \Big\langle \nabla V_{\mu^*}(x^*(s)), \frac{1}{\lambda^*} x^{*\prime}(s)\Big\rangle\, ds \\ &= \frac{1}{\lambda^*}\Big(V_{\mu^*}\big(x^*(T)\big) - V_{\mu^*}\big(x^*(0)\big)\Big) = 0, \end{aligned}$$

giving a contradiction.

If $\lambda^* = 0$ then $(x^*)' = 0$. Therefore $x^* \equiv a$ for some $a \in \mathbb{R}^n, |a| = r$. According to Definition 8, for every $f \in \mathcal{P}_{\mathcal{F}}(a, \mu^*)$ we have

$$\int_0^T \big\langle \nabla V_{\mu^*}(a), f(s)\big\rangle\, ds = \Big\langle \nabla V_{\mu^*}(a), \int_0^T f(s)\, ds\Big\rangle = T\big\langle \nabla V_{\mu^*}(a), \Pi_L f\big\rangle > 0. \tag{3.6}$$

Consequently, $\Pi_L f \neq 0$ for all $f \in \mathcal{P}_{\mathcal{F}}(a, \mu^*)$, in particular, $\Pi_L f^* \neq 0$. But $\Pi_L f^* = \Pi_L f_0^* = 0$. That is a contradiction.

Thus multimap Σ is a homotopy on $\partial B_{r,\varepsilon}$ connecting the multimaps $\Sigma(x,\mu,1) = G_r(x,\mu)$ and

$$\Sigma(x,\mu,0) = \{P_L x + \Pi_L \mathcal{P}_{\mathcal{F}}(x,\mu)\} \times \{|\mu|^2 - \varepsilon^2\}.$$

By virtue of the homotopy invariance of the coincidence index we obtain

$$Ind(\ell, G_r, B_{r,\varepsilon}) = Ind(\ell, \Sigma(\cdot,\cdot,0), B_{r,\varepsilon}).$$

The multimap $P_L + \Pi_L \mathcal{P}_{\mathcal{F}}$ takes values in $\mathbb{R}^n$, so applying the restriction property of the coincidence degree we have

$$Ind(\ell, \Sigma(\cdot,\cdot,0), B_{r,\varepsilon}) = Ind(\ell, \Sigma(\cdot,\cdot,0), \overline{U}_{r,\varepsilon}),$$

where $\overline{U}_{r,\varepsilon} = B_{r,\varepsilon} \cap \mathbb{R}^{n+k}$.

In the space $\mathbb{R}^{n+1}$ the vector field $\ell - \Sigma(\cdot,\cdot,0)$ has the form

$$\ell(y,\mu) - \Sigma(y,\mu,0) = \{-\Pi_L \mathcal{P}_{\mathcal{F}}(y,\mu)\} \times \{\varepsilon^2 - |\mu|^2\}, \quad \forall (y,\mu) \in \overline{U}_{r,\varepsilon}.$$

Consider now the multimap: $\Gamma\colon \overline{U}_{r,\varepsilon} \times [0,1] \to Kv(\mathbb{R}^{n+1})$ defined as

$$\Gamma(y,\mu,\lambda) = \{-\lambda \Pi_L \mathcal{P}_F(y,\mu) + (\lambda - 1)\nabla V_\mu(y)\} \times \{\varepsilon^2 - |\mu|^2\}.$$

It is clear that Γ is a compact u.s.c. multimap. Assume that there exists $(y,\mu,\lambda) \in \partial U_{r,\varepsilon} \times [0,1]$ such that $0 \in \Gamma(y,\mu,\lambda)$. Then we obtain

$$\begin{cases} |\mu| = \varepsilon \\ (\lambda - 1)\nabla V_\mu(y) \in \lambda \Pi_L \mathcal{P}_{\mathcal{F}}(y,\mu), \end{cases}$$

and by virtue of (3.6) we get the contradiction. So, Γ is a homotopy connecting $\ell - \Sigma(\cdot,\cdot,0)$ and $\widetilde{V}_\varepsilon$, therefore, by assumption

$$Ind(\ell, \Sigma(\cdot,\cdot,0), \overline{U}_{r,\varepsilon}) = deg(\widetilde{V}_\varepsilon, \overline{U}_{r,\varepsilon}) = deg(\widetilde{V}_\varepsilon, \overline{O_\varepsilon}) = ind\, V_\mu \neq 0. \tag{3.7}$$

Step 4. Let $\mathcal{O} \subset \mathcal{C} \times \mathbb{R}^k$ be an open set defined as

$$\mathcal{O} = (\mathcal{C} \times \mathbb{R}^k) \setminus (\{0\} \times (\mathbb{R}^k \setminus B^k(0,\varepsilon_0))).$$

From $Ind\,(\ell, G_r, B_{r,\varepsilon}) \neq 0$ for every $\varepsilon \in (0,\varepsilon_0)$ and for all $r \in (0,\delta_\varepsilon)$ it follows that there exists $(x,\mu) \in B_{r,\varepsilon}$ such that $\ell(x,\mu) \in G_r(x,\mu)$, or equivalently

$$\begin{cases} x \in G(x,\mu), \\ \|x\|_{\mathcal{C}} = r, \end{cases}$$

i.e., $(x,\mu) \in B_{r,\varepsilon}$ is a nontrivial solution to problem (3.2). Therefore, $(0,0)$ is a bifurcation point of problem (3.2), and hence, it is a bifurcation point of problem (3.1). Denote by $\mathcal{R} \subset \mathcal{S} \cup \{(0,0)\} \subset \mathcal{O}$ the connected component of $(0,0)$. Let us demonstrate that $\mathcal{R}$ is a non-compact component. Assume to the contrary that $\mathcal{R}$ is compact. Then there exists an open bounded subset $U \subset \mathcal{O}$ such that

$$\overline{U} \subset \mathcal{O}, \qquad \mathcal{R} \subset U \quad \text{and} \quad \partial U \cap \mathcal{S} = \emptyset.$$

Hence, for each $r > 0$

$$\ell(x,\mu) \notin G_r(x,\mu), \quad \forall (x,\mu) \in \partial U.$$

Further, for any $0 < r < R$, the compact u.s.c. multimaps G_r and G_R on $\overline{U}$ can be joined by the homotopy $G_{\lambda r + (1-\lambda)R}$. For sufficiently large R,

$$\ell(x,\mu) \notin G_R(x,\mu), \quad \forall (x,\mu) \in \overline{U},$$

so, $Ind\,(\ell, G_R, \overline{U}) = 0$. Therefore, $Ind\,(\ell, G_r, \overline{U}) = 0$ for all $r > 0$.

Now, let $\Lambda = \{\mu \in \mathbb{R}^k : (0, \mu) \in \overline{U}\}$. From $\overline{U} \subset \mathcal{O}$ it follows that

$$\Lambda \subset B^k(0, \varepsilon_0). \tag{3.8}$$

From Lemma 4(a) and the continuous dependence of the number δ_ε on ε it follows that we can choose $0 < \varepsilon < \varepsilon_0$ and $0 < r < \delta_\varepsilon$ such that $B_{r,\varepsilon} \subset U$ and inclusion

$$x \in G(x, \mu)$$

has only trivial solutions in the ball $\overline{B_{\mathcal{C}}(0, r)}$ for all $\mu \in \mathbb{R}^k : \varepsilon \leq |\mu| < \varepsilon_0$.

From (3.8) and the choice of r, ε (we can take r, ε sufficiently small) we have

$$Coin(\ell, G_r, \overline{U}) := \{(x, \mu) \in \overline{U} : \ell(x, \mu) \in G_r(x, \mu)\} \subset B_{r,\varepsilon}.$$

So, we obtain

$$0 = Ind\,(\ell, G_r, \overline{U}) = Ind\,(\ell, G_r, B_{r,\varepsilon}) \neq 0,$$

that is the contradiction. Thus, $\mathcal{R}$ is a non-compact component, i.e., either $\mathcal{R}$ is unbounded or $\overline{\mathcal{R}} \cap \partial\overline{\mathcal{O}} \neq \emptyset$. ■

4. Main result

In this section we will use the symbols $\mathcal{W}_T^{1,2}, \mathcal{C}$ defined in Section 2.1 with a remark that $n = 2$. Now we study the global bifurcation of solutions of problem (1.1). Assume that

(A1) for each $(t, z, \mu) \in [0, T] \times \mathbb{R}^2 \times \mathbb{R}^2$ the set

$$f(t, z, \Omega, \mu) := \{f(t, z, y, \mu) : y \in \Omega\}$$

is convex for every convex subset $\Omega \subset \mathbb{R}^m$;

(A2) there exist positive numbers $\alpha_f, \alpha_G, c_1, c_2$ such that $\alpha_f < c$ and

$$|f(t, z, y, \mu)| \leq \alpha_f\,|z|\,(|y| + |\mu|),$$
$$|G(t, z, \mu)| \leq \alpha_G\,|z|^{c_1}(1 + |\mu|^{c_2}),$$

for $(t, z, y, \mu) \in [0, T] \times \mathbb{R}^2 \times \mathbb{R}^m \times \mathbb{R}^2$, where $c > 0$ is the given number in (1.2);

(A3) F is monotone on K and there exists $a > 0$ such that

$$\big\langle w, F(w)\big\rangle \geq a\,|w|^2 \quad \text{for all } w \in K.$$

Lemma 5 (*Cf. Proposition 6.2 [33]*). *Let condition* (A3) *holds. Then*

(a) *for every $r \in \mathbb{R}^m$ the solution set*

$$SOL(K, r + F) = \big\{w \in K : \big\langle \widetilde{w} - w, r + F(w)\big\rangle \geq 0,\ \forall \widetilde{w} \in K\big\}$$

is nonempty convex and closed;

(b) *$|w| \leq \frac{1}{a}|r|$ for all $w \in SOL(K, r + F), r \in \mathbb{R}^m$, where a is the constant from condition* (A3)*.*

Proof. (a) It is clear that for every $r \in \mathbb{R}^m$ the map $r + F$ is monotone on K. From (A3) it follows that

$$\frac{\big\langle w, r + F(w)\big\rangle}{|w|} \to \infty \quad \text{as } \|w\| \to \infty \text{ and } w \in K.$$

Hence, by virtue of Lemma 3 the set $SOL(K, r+F)$ is nonempty. Since F is continuous the set $SOL(K, r+F)$ is closed. Let us show that it is convex, i.e., we need to show that for $y, z \in SOL(K, r + F)$ and $\lambda \in [0, 1]$

$$\big\langle u - \lambda y - (1-\lambda)z, r + F(\lambda y + (1-\lambda)z)\big\rangle \geq 0 \quad \text{for all } u \in K,$$

or equivalently (see Lemma 1):

$$\big\langle u - \lambda y - (1-\lambda)z, r + F(u)\big\rangle \geq 0 \quad \text{for all } u \in K.$$

In fact, since $y, z \in SOL(K, r + F)$ we have

$$\big\langle u - y, r + F(y)\big\rangle \geq 0 \quad \text{for all } u \in K,$$

and

$$\big\langle u - z, r + F(z)\big\rangle \geq 0 \quad \text{for all } u \in K,$$

or equivalently (see Lemma 1):

$$\big\langle u - y, r + F(u)\big\rangle \geq 0 \quad \text{for all } u \in K,$$

and

$$\big\langle u - z, r + F(u)\big\rangle \geq 0 \quad \text{for all } u \in K.$$

Consequently,

$$\big\langle u - \lambda y - (1-\lambda)z, r + F(u)\big\rangle = \lambda\big\langle u - y, r + F(u)\big\rangle + (1-\lambda)\big\langle u - z, r + F(u)\big\rangle \geq 0$$

for all $u \in K$.

(b) Now, for $w \in SOL(K, r + F)$, by virtue of Lemma 2 we have $\big\langle w, r + F(w)\big\rangle = 0$. Therefore,

$$\big|\big\langle w, r\big\rangle\big| = \big|\big\langle w, F(w)\big\rangle\big|.$$

Applying (A3) we obtain

$$a|w|^2 \leq \big|\big\langle w, r\big\rangle\big| \leq |w|\,|r|.$$

So, $|w| \leq \frac{1}{a}|r|$. ■

Define the multimap $U \colon [0, T] \times \mathbb{R}^2 \times \mathbb{R}^2 \to Cv(K)$,

$$U(t, z, \mu) = SOL(K, G(t, z, \mu) + F),$$

and $\Phi \colon [0, T] \times \mathbb{R}^2 \times \mathbb{R}^2 \to Kv(\mathbb{R}^2)$

$$\Phi(t, z, \mu) = \big\{A(\mu)z + f(t, z, y, \mu) \colon y \in U(t, z, \mu)\big\}.$$

Lemma 6. *Let conditions* (A1)–(A3) *hold. Then Φ satisfies conditions* ($\mathcal{F}1$)–($\mathcal{F}4$) *with a remark that $n = k = 2$.*

Proof. From (A2) and Lemma 5(b) it follows that

$$\|U(t, z, \mu)\| \leq \frac{1}{a}\,|G(t, z, \mu)| \leq \frac{\alpha_G}{a}\,|z|^{c_1}(1 + |\mu|^{c_2}) \tag{4.1}$$

for $(t, z, \mu) \in [0, T] \times \mathbb{R}^2 \times \mathbb{R}^2$. Therefore, the restriction $U_{|\Omega}$ of multimap U on any bounded subset $\Omega \subset [0, T] \times \mathbb{R}^2 \times \mathbb{R}^2$ is compact. By virtue of Lemma 5(a) U is closed multimap, and hence, it is u.s.c. Since the maps A and f are continuous the multimap Φ is u.s.c., too. So, Φ satisfies conditions $(\mathcal{F}1)$–$(\mathcal{F}2)$. Conditions $(\mathcal{F}3)$–$(\mathcal{F}4)$ follows immediately from (A2) and (4.1). ∎

Now, by using Filippov's Implicit Function Lemma (see, e.g. Theorem 1.5.15 [6] or Theorem 1.3.3 [28]) we can substitute problem (1.1) by the following equivalent problem

$$\begin{cases} x'(t) \in \Phi(t, x(t), \mu) & \text{for a.e. } t \in [0, T], \\ x(0) = x(T). \end{cases} \tag{4.2}$$

Definition 9. By a solution to problem (1.1) we mean a triplet (x, u, μ) consisting of a function $x \in \mathcal{W}_T^{1,2}$, an integrable function $u\colon [0, T] \to K$ and a vector $\mu \in \mathbb{R}^2$ that satisfies (1.1), or equivalently, by a solution to problem (1.1) we mean a pair $(x, \mu) \in \mathcal{W}_T^{1,2} \times \mathbb{R}^2$ that satisfies (4.2).

From (A2) it follows that $(0, \mu)$ is the trivial solution to (1.1) for all $\mu \in \mathbb{R}^2$. Let us denote by $\mathcal{S}$ the set of all nontrivial solutions to (1.1).

Theorem 2. *Let conditions* (A1)–(A3) *hold. Then* $(0, 0)$ *is a unique bifurcation point for solutions of* (1.1) *and, moreover, there is an unbounded subset* $\mathcal{R} \subset \mathcal{S}$ *such that* $(0, 0) \in \overline{\mathcal{R}}$.

Proof. Firstly, let us show that the family of functions $V_\mu\colon \mathbb{R}^2 \to \mathbb{R}$,

$$V_\mu(w) = \frac{c}{2}\big(\mu_1 w_1^2 + 2\mu_2 w_1 w_2 - \mu_1 w_2^2\big), \quad w = (w_1, w_2),\ \mu = (\mu_1, \mu_2),$$

is a family of local integral guiding functions for (4.2) at $(0, 0)$. In fact, let $\varepsilon > 0$ and $\mu \in S^1(0, \varepsilon)$ be arbitrary. For $x \in \mathcal{C}$ take any $g \in \mathcal{P}_\Phi(x, \mu)$ ($\mathcal{P}_\Phi$ is defined similarly $\mathcal{P}_\mathcal{F}$). Then there is an integrable function $u\colon [0, T] \to K$ such that

$$g(t) = A(\mu)x(t) + f\big(t, x(t), u(t), \mu\big) \quad \text{for a.e. } t \in [0, T].$$

By virtue of (A2) and (4.1) we have

$$\begin{aligned} \int_0^T \langle \nabla V_\mu(x(t)), g(t)\rangle dt &= \int_0^T \Big\langle A(\mu)x(t), A(\mu)x(t) + f\big(t, x(t), u(t), \mu\big)\Big\rangle dt \\ &\geq \int_0^T |A(\mu)x(t)|^2 dt - \int_0^T |A(\mu)x(t)|\,|f(t, x(t), u(t), \mu)| dt \\ &\geq c^2\,|\mu|^2\,\|x\|_2^2 - c|\mu| \int_0^T |x(t)|\,\alpha_f |x(t)|(|u(t)| + |\mu|) dt \\ &\geq c^2\,|\mu|^2\,\|x\|_2^2 - c\alpha_f |\mu|^2 \|x\|_2^2 - c\alpha_f |\mu| \int_0^T |x(t)|^2\,\frac{\alpha_G}{a}\,|x(t)|^{c_1}(1 + |\mu|^{c_2}) dt \\ &\geq c(c - \alpha_f)\,|\mu|^2\,\|x\|_2^2 - \frac{c}{a}\,\alpha_f \alpha_G |\mu|(1 + |\mu|^{c_2})\|x\|_\mathcal{C}^{c_1}\,\|x\|_2^2. \end{aligned}$$

Therefore,

$$\int_0^T \langle \nabla V_\mu(x(t)), g(t)\rangle dt > 0$$

provided

$$0 < \|x\|_\mathcal{C} < \sqrt[c_1]{\frac{a(c - \alpha_f)|\mu|}{\alpha_f \alpha_G (1 + |\mu|^{c_2})}} = \sqrt[c_1]{\frac{a(c - \alpha_f)\varepsilon}{\alpha_f \alpha_G (1 + \varepsilon^{c_2})}}.$$

Put

$$\delta_\varepsilon = \frac{1}{2}\sqrt[c_1]{\frac{a(c-\alpha_f)\varepsilon}{\alpha_f\alpha_G(1+\varepsilon^{c_2})}}.$$

Then δ_ε continuously depends on ε and

$$\int_0^T \langle \nabla V_\mu(x(t)), g(t)\rangle dt > 0 \quad \text{provided} \quad 0 < \|x\|_{\mathcal{C}} \le \delta_\varepsilon. \tag{4.3}$$

Thus, V_μ is a family of local integral guiding functions for problem (4.2) at $(0,0)$.

For each $\varepsilon > 0$, choose δ_ε as above. Set $O_\varepsilon = B^4\big(0, \sqrt{\varepsilon^2+\delta_\varepsilon^2}\,\big)$ and consider the map

$$\begin{aligned} &\widetilde{V}_\varepsilon\colon \overline{O_\varepsilon} \to \mathbb{R}^3,\\ &\widetilde{V}_\varepsilon(w,\mu) \;=\; \big\{-\nabla V_\mu(w)\big\}\times\big\{\varepsilon^2-|\mu|^2\big\}\\ &\qquad\qquad\;=\; \big\{-(2\mu_1w_1+2\mu_2w_2), -(2\mu_2w_1-2\mu_1w_2), \varepsilon^2-|\mu|^2\big\}. \end{aligned}$$

Let us show that $ind\, V_\mu \neq 0$. Toward this goal, consider the following continuous map

$$\begin{aligned} &H\colon \overline{O_\varepsilon}\times[0,1]\to\mathbb{R}^3,\\ &H(w,\mu,\lambda) = \{-\nabla V_\mu(w)\}\times\{\lambda|w|^2+(1-\lambda)\varepsilon^2-|\mu|^2\}. \end{aligned}$$

Assume that there exists $(w,\mu,\lambda)\in\partial O_\varepsilon\times[0,1]$ such that $H(w,\mu,\lambda)=0$, then we have

$$\begin{cases} -\nabla V_\mu(w) = 0,\\ \lambda|w|^2-|\mu|^2 = (\lambda-1)\varepsilon^2,\\ |w|^2+|\mu|^2 = \varepsilon^2+\delta_\varepsilon^2. \end{cases}$$

From the second and third equations of the above system it follows that

$$|w|^2 = \frac{\lambda\varepsilon^2+\delta_\varepsilon^2}{1+\lambda} \quad\text{and}\quad |\mu|^2 = \frac{\varepsilon^2+\lambda\delta_\varepsilon^2}{1+\lambda}.$$

Therefore, w and μ are non-zero elements in $\mathbb{R}^2$. That contradicts to the first equation of the system. Thus, H is a homotopy connecting the maps $\widetilde{V}_\varepsilon = H(\cdot,\cdot,0)$ and $H(\cdot,\cdot,1)$. By the homotopy invariance property of the topological degree we obtain

$$deg(\widetilde{V}_\varepsilon, \overline{O_\varepsilon}) = deg\big(H(\cdot,\cdot,1), \overline{O_\varepsilon}\big).$$

On the other hand, the map $H(\cdot,\cdot,1)\colon \mathbb{R}^4\to\mathbb{R}^3$ vanishes only at $(0,0)$ and the restriction $h = -H(\cdot,\cdot,1)_{|_{S^3}}\colon S^3\to S^2$ is the Hopf fibration (see, e.g. [27]). Hence,

$$deg\big(H(\cdot,\cdot,1),\overline{O_\varepsilon}\big) = deg\big(H(\cdot,\cdot,1),\overline{B^4}\big) = deg(-h,\overline{B^4}) = \varSigma[-h]\neq 0,$$

where $\varSigma$ is defined in Example 1.

Therefore, $(0,0)$ is a bifurcation point for solutions of (4.2). Moreover, from the fact that relation (4.3) holds true for all $\varepsilon>0$ it follows that $(0,0)$ is the unique bifurcation point for solutions of (4.2). Now, the application of Theorem 1 ends the proof. ∎

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