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RESEARCH PAPER

DECAY SOLUTIONS FOR A CLASS OF FRACTIONAL DIFFERENTIAL VARIATIONAL INEQUALITIES

Tran Dinh Ke [1], Nguyen Van Loi [2], Valeri Obukhovskii [3]

Abstract

Our aim is to study a new class of differential variational inequalities involving fractional derivatives. Using the fixed point approach, the existence of decay solutions to the mentioned problem is proved.

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Key Words and Phrases: differential variational inequality, fixed point, measure of noncompactness, MNC estimate, functional differential inclusion, condensing operator

1. Introduction

We consider the differential variational inequality (DVI) of the following form

$$ {}^C D_0^\alpha x(t) = Ax(t) + B(t, x(t), x_t)u(t), t \in J := [0, T], \tag{1.1}$$

$$ \langle v - u(t), F(t, x(t)) + G(u(t))\rangle \geq 0, \forall v \in K, \text{ for a.e. } t \in J, \tag{1.2}$$

$$ x(s) + h(x)(s) = \varphi(s), s \in [-\tau, 0], \tag{1.3}$$

where $x(t) \in \mathbb{R}^n, u(t) \in K$ with K being a closed convex subset in $\mathbb{R}^m$. Here ${}^C D_0^\alpha, \alpha \in (0,1)$, denotes the Caputo derivative of fractional order α, x_t stands for the history of state function up to the time t, i.e. $x_t(\theta) =$

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$x(t+\theta)$, $\theta \in [-\tau, 0]$; $\langle \cdot, \cdot \rangle$ is the inner product on $\mathbb{R}^m$, A, B, F, G and h are given maps which will be specified in Section 3.

The notion of differential variational inequality was firstly used by Aubin and Cellina [1] in 1984. In their book the authors considered the problem:

$$\begin{cases} \forall t \geq 0,\ x(t) \in K, \\ \sup_{y \in K} \big\langle x'(t) - f(x(t)), x(t) - y \big\rangle = 0, \\ x(0) = x_0, \end{cases} \tag{1.4}$$

where K is a convex closed subset. The problem (1.4) was replaced with the differential inclusion of the form:

$$\begin{cases} x'(t) \in F(t, x(t)), \\ x(0) = x_0. \end{cases}$$

Then the solvability of (1.4) can be studied by the topological tools of multivalued analysis. After this work, the theory of DVIs was considered and expanded in the work of Avgerinous and Papageorgiou [4] in 1997. Moreover, Avgerinous and Papageorgiou studied the periodic solutions to the DVI of the form:

$$\begin{cases} -x'(t) \in N_{K(t)}\big(x(t)\big) + F\big(t, x(t)\big) \text{ for a.e. } t \in [0, b] \\ x(0) = x(b), \end{cases} \tag{1.5}$$

where $N_{K(t)}\big(x(t)\big)$ denotes the normal cone of the convex closed set $K(t)$ at the point $x(t)$.

However, DVIs were first systematically studied by Pang and Stewart [25]. DVIs are useful for representing models involving both dynamics and constraints in the form of inequalities which arise in many applied problems, for example, mechanical impact problems, electrical circuits with ideal diodes, Coulomb friction problems for contacting bodies, economical dynamics and related problems such as dynamic traffic networks. Some existence results for DVIs can be found in [14, 15, 22] (see also the references therein).

In the last few years, the theory of fractional differential equations (FrDEs) has attracted much attention. The FrDEs were proved to be an effective tool to model realistic problems in fluid flow, rheology, electrical networks, viscoelasticity, electrochemistry, etc. For complete references, we send to some significant works, e.g., the monographs of Kilbas et al. [20], Kiryakova [21], Miller and Ross [23] and Podlubny [26]. As for nonlinear FrDEs, there have been many studies aimed to investigate the problems of solvability, controllability and optimal control. Some recent results in these directions can be found in [3, 6, 7, 8, 13, 19, 24].

As a matter of fact, system (1.1)-(1.3) can be seen as a control problem subject to constraints. Though many contributions for FrDEs have been carried out, up to our knowledge, no attempt has been made to study fractional DVIs appeared as (1.1)-(1.3). In the present paper, by combining the topological methods and the fractional calculus we study the existence of solutions to problem (1.1)-(1.3) and the existence of decay solutions to this problem when J is an infinite interval. Regarding the last objective, we find a suitable space of solutions and define a regular measure of noncompactness on this space. This construction enables us to apply the fixed point theory for condensing multivalued maps to obtain a compact set of decay solutions $x(\cdot)$ with polynomial decay rate, that is $t^\alpha \|x(t)\| = O(1)$ as $t \to \infty$.

The paper is organized in the following way. In Section 2 we recall some notions and facts from the multivalued analysis and fractional calculus. Section 3 deals with the existence of solutions to (1.1)-(1.3) on compact interval and in Section 4 the existence of decay solutions is proved.

2. **Preliminaries**

2.1. **Measure of noncompactness and multivalued maps.** Let E be a Banach space. Denote

$$\mathcal{P}(E) = \{B \subset E : B \neq \emptyset\},$$
$$\mathcal{B}(E) = \{B \in \mathcal{P}(E) : B \text{ is bounded}\}.$$

We will use the following definition of the measure of noncompactness (see, e.g., [2, 18]).

Definition 2.1. A function $\beta : \mathcal{B}(E) \to \mathbb{R}^+$ is called *a measure of noncompactness* (MNC) in E if

$$\beta(\overline{co}\,\Omega) = \beta(\Omega) \text{ for every } \Omega \in \mathcal{B}(E),$$

where $\overline{co}\,\Omega$ is the closure of the convex hull of Ω. An MNC β is called

i) monotone if $\Omega_0, \Omega_1 \in \mathcal{B}(E),\ \Omega_0 \subset \Omega_1$ implies $\beta(\Omega_0) \le \beta(\Omega_1)$;
ii) nonsingular if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for any $a \in E, \Omega \in \mathcal{B}(E)$;
iii) invariant with respect to union with compact set if $\beta(K \cup \Omega) = \beta(\Omega)$ for every relatively compact set $K \subset E$ and $\Omega \in \mathcal{B}(E)$;
iv) algebraically semi-additive if $\beta(\Omega_0 + \Omega_1) \le \beta(\Omega_0) + \beta(\Omega_1)$ for any $\Omega_0, \Omega_1 \in \mathcal{B}(E)$;
v) regular if $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω.

An important example of MNC is *the Hausdorff* MNC $\chi(\cdot)$, which is defined as

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \ \Omega \text{ has a finite } \varepsilon\text{-net}\}.$$

For a fixed $\theta \in \mathbb{R}$ and $T > \theta$, it is known that the Hausdorff MNC on the space $C([\theta, T]; \mathbb{R}^n)$ is given by

$$\chi_T(D) = \frac{1}{2} \lim_{\delta \to 0} \sup_{x \in D} \max_{t,s \in [\theta,T], |t-s|<\delta} \|x(t) - x(s)\|, \tag{2.1}$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$ (see [2]). This quantity is called the modulus of equicontinuity of $D \subset C([\theta, T]; \mathbb{R}^n)$.

Consider the space $BC(\theta, \infty; \mathbb{R}^n)$ of bounded continuous functions on $[\theta, \infty)$ taking values in $\mathbb{R}^n$. Denote by π_T the restriction operator on this space, that is $\pi_T(x)$ is the restriction of x on $[\theta, T]$. Then the function χ_∞ defined by

$$\chi_\infty(D) = \sup_{T > \theta} \chi_T(\pi_T(D)), \quad D \subset BC(\theta, \infty; \mathbb{R}^n), \tag{2.2}$$

is an MNC. One can check that this MNC satisfies all properties given in Definition 2.1, with the exception of regularity. Indeed, we will testify this claim by choosing the sequence $\{f_k\} \subset BC(0, \infty; \mathbb{R})$ as follows

$$f_k(t) = \begin{cases} 0, & t \notin [k, k+1], \\ 2t - 2k, & t \in [k, k + \frac{1}{2}], \\ -2t + 2k + 2, & t \in [k + \frac{1}{2}, k+1]. \end{cases}$$

Then it is obvious that $\{\pi_T(f_k)\}$ is compact (converging to 0 in $C([0, T]; \mathbb{R})$) for any $T > 0$. However, one sees that

$$\sup_{t \geq 0} |f_k(t) - f_l(t)| = 1 \text{ for } k \neq l,$$

and then $\{f_k\}$ is not a Cauchy sequence in $BC(0, \infty; \mathbb{R})$. This fact tells us that $\chi_T(\pi_T(\{f_k\})) = 0$ for any $T > 0$ and then $\chi_\infty(\{f_k\}) = 0$, but $\{f_k\}$ is non-compact.

We also make use of the following MNCs on $BC(\theta, \infty; \mathbb{R}^n)$ (see [5]):

$$d_T(D) = \sup_{x \in D} \sup_{t \geq T} \|x(t)\|, \tag{2.3}$$

$$d_\infty(D) = \lim_{T \to \infty} d_T(D). \tag{2.4}$$

Let

$$\chi^*(D) = \chi_\infty(D) + d_\infty(D). \tag{2.5}$$

Then χ^* is an MNC on $BC(\theta, \infty; \mathbb{R}^n)$. We now prove its regularity.

Lemma 2.1. *The MNC χ^* defined by* (2.5) *is regular.*

P r o o f. Let $D \subset BC(\theta,\infty;\mathbb{R}^n)$ be a bounded set such that $\chi^*(D) = 0$. We show that D is relatively compact. Let $PBC(\theta,\infty;\mathbb{R}^n)$ be the space of piecewise continuous and bounded functions on $\mathbb{R}^+$, taking values in $\mathbb{R}^n$. This is a Banach space with the norm

$$\|x\|_{PBC} = \sup_{t\geq\theta} \|x(t)\|,$$

and contains $BC(\theta,\infty;\mathbb{R}^n)$ as a closed subspace.

For $\epsilon > 0$, since $d_\infty(D) = 0$ one can choose $T > 0$ such that $\sup_{t\geq T}\|x(t)\| < \frac{\epsilon}{2}, \forall x \in D$. This means that

$$\|x - \pi_T(x)\|_{PBC} < \frac{\epsilon}{2}, \forall x \in D,$$

here $\pi_T(x)$ agrees with a function in $PBC(\theta,\infty;\mathbb{R}^n)$ in the following manner

$$\pi_T(x) = \begin{cases} x(t), & t \in [\theta, T], \\ 0, & t > T. \end{cases}$$

Now since $\pi_T(D)$ is a compact set in $C([\theta,T];\mathbb{R}^n)$, we can write

$$\pi_T(D) \subset \bigcup_{i=1}^{N} B_T(x_i, \frac{\epsilon}{2}) \tag{2.6}$$

where $x_i \in C([\theta,T];\mathbb{R}^n), i = 1,...,N$, the notation $B_T(x,r)$ stands for the ball in $C([\theta,T];\mathbb{R}^n)$ centered at x with radius r. Put

$$\hat{x}_i(t) = \begin{cases} x_i(t), & t \in [\theta, T], \\ 0, & t > T, \end{cases}$$

then $\{\hat{x}_i\}_{i=1}^N$ belong to $PBC(\theta,\infty;\mathbb{R}^n)$. We now assert that

$$D \subset \bigcup_{i=1}^{N} B_\infty(\hat{x}_i, \epsilon),$$

here $B_\infty(x,r)$ is the ball in $PBC(\theta,\infty;\mathbb{R}^n)$ with center x and radius r. Indeed, if $x \in D$ then by (2.6) there is a number $k \in \{1,...,N\}$ such that

$$\|\pi_T(x) - x_k\|_C < \frac{\epsilon}{2},$$

here $\|\cdot\|_C$ is the norm in $C([\theta,T];\mathbb{R}^n)$. This implies

$$\|\pi_T(x) - \hat{x}_k\|_{PBC} < \frac{\epsilon}{2}.$$

Then

$$\begin{aligned} \|x - \hat{x}_k\|_{PBC} &\leq \|x - \pi_T(x)\|_{PBC} + \|\pi_T(x) - \hat{x}_k\|_{PBC} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Thus $x \in B_\infty(\hat{x}_k, \epsilon)$. We have $D \subset \bigcup\limits_{i=1}^{N} B_\infty(\hat{x}_i, \epsilon)$, and hence D is relatively compact in $PBC(\theta,\infty;\mathbb{R}^n)$. Since $BC(\theta,\infty;\mathbb{R}^n)$ and $PBC(\theta,\infty;\mathbb{R}^n)$

have the same norm, we conclude that D is a relatively compact set in $BC(\theta, \infty; \mathbb{R}^n)$. The proof is complete. □

In the sequel, we make use of some notions and facts of multivalued analysis. Let Y be a metric space.

Definition 2.2. A multivalued map (multimap) $\mathcal{F} : Y \to \mathcal{P}(E)$ is said to be:

i) upper semicontinuous (u.s.c) if $\mathcal{F}^{-1}(V) = \{y \in Y : \mathcal{F}(y) \cap V \neq \emptyset\}$ is a closed subset of Y for every closed set $V \subset E$;
ii) weakly upper semicontinuous (weakly u.s.c) if $\mathcal{F}^{-1}(V)$ is a closed subset of Y for all weakly closed sets $V \subset E$;
iii) closed if its graph $\Gamma_{\mathcal{F}} = \{(y, z) : y \in Y, z \in \mathcal{F}(y)\}$ is a closed subset of $Y \times E$;
iv) compact if $\mathcal{F}(Y)$ is relatively compact in E;
v) quasicompact if its restriction to any compact subset $A \subset Y$ is compact.

Lemma 2.2 ([18, Theorem 1.1.12]). *Let $G : Y \to \mathcal{P}(E)$ be a closed quasicompact multimap with compact values. Then G is u.s.c.*

Lemma 2.3 ([9, Proposition 2]). *Let X be a Banach space and Ω be a nonempty subset of another Banach space. Assume that $\mathcal{G} : \Omega \to \mathcal{P}(X)$ is a multimap with weakly compact, convex values. Then $\mathcal{G}$ is weakly u.s.c iff $\{x_n\} \subset \Omega$ with $x_n \to x_0 \in \Omega$ and $y_n \in \mathcal{G}(x_n)$ implies $y_n \rightharpoonup y_0 \in \mathcal{G}(x_0)$, up to a subsequence.*

Definition 2.3. A multimap $\mathcal{F} : Z \subseteq E \to \mathcal{P}(E)$ is said to be condensing with respect to an MNC β (β-condensing) if for any bounded set $\Omega \subset Z$, the relation

$$\beta(\Omega) \leq \beta(\mathcal{F}(\Omega))$$

implies the relative compactness of Ω.

Let β be a monotone nonsingular MNC in E. The application of the topological degree theory for condensing multimaps (see, e.g., [18]) yields the following fixed point principle, which will be used to prove the existence of solutions to (1.1)-(1.3).

Theorem 2.1. [18, Corollary 3.3.1] *Let $\mathcal{M}$ be a bounded convex closed subset of E and let $\mathcal{F} : \mathcal{M} \to \mathcal{P}(\mathcal{M})$ be a u.s.c and β-condensing multimap* *with compact convex values. Then the fixed point set* $\mathrm{Fix}(\mathcal{F}) := \{x = \mathcal{F}(x)\}$ *is a nonempty compact set.*

2.2. **Fractional calculus.** Let $L^1(0,T;\mathbb{R}^n)$ be the space of integrable functions on $[0,T]$, taking values in $\mathbb{R}^n$.

DEFINITION 2.4. The fractional integral of order $\alpha > 0$ of a function $f \in L^1(0,T;\mathbb{R}^n)$ is defined by

$$I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds,$$

where Γ is the Gamma function, provided the integral converges.

DEFINITION 2.5. For a function $f \in C^N([0,T];\mathbb{R}^n)$, the Caputo fractional derivative of order $\alpha \in (N-1,N)$ is defined by

$${}^C D_0^\alpha f(t) = \frac{1}{\Gamma(N-\alpha)} \int_0^t (t-s)^{N-\alpha-1} f^{(N)}(s)ds.$$

Let $S(t) = e^{tA}, t \geq 0$, be the semigroup generated by A. Put

$$S_\alpha(t) = E_{\alpha,1}(t^\alpha A), \tag{2.7}$$

$$P_\alpha(t) = E_{\alpha,\alpha}(t^\alpha A), \tag{2.8}$$

where $E_{\alpha,\beta}$ is the Mittag-Leffler function defined by

$$E_{\alpha,\beta}(t) = \sum_{k=0}^\infty \frac{t^k}{\Gamma(\alpha k + \beta)}.$$

Then by [29] we have the representation for S_α and P_α as follows:

$$S_\alpha(t)z = \int_0^\infty \phi_\alpha(\theta) S(t^\alpha \theta) z d\theta, \tag{2.9}$$

$$P_\alpha(t)z = \alpha \int_0^\infty \theta \phi_\alpha(\theta) S(t^\alpha \theta) z d\theta, z \in \mathbb{R}^n, \tag{2.10}$$

where ϕ_α is a probability density function defined on $(0,\infty)$, that is, $\phi_\alpha(\theta) \geq 0$ and $\int_0^\infty \phi_\alpha(\theta)d\theta = 1$. Moreover, ϕ_α has the expression

$$\phi_\alpha(\theta) = \frac{1}{\pi} \sum_{n=1}^\infty \frac{(-\theta)^n}{(n-1)!} \Gamma(n\alpha) \sin(n\pi\alpha).$$

PROPOSITION 2.1. *If the C_0-semigroup $\{S(t)\}_{t\geq 0}$ generated by A is exponentially stable, i.e., there are positive numbers a, M such that*

$$\|e^{tA}\| \leq M e^{-at},$$

then $\|S_\alpha(t)\|, \|P_\alpha(t)\| = O(t^{-\alpha})$ *as* $t \to \infty$.

P r o o f. By the fact that (see, e.g. [28])

$$\int_0^\infty \phi_\alpha(\theta) e^{-z\theta} d\theta = E_{\alpha,1}(-z),$$
$$\int_0^\infty \alpha\theta\phi_\alpha(\theta) e^{-z\theta} d\theta = E_{\alpha,\alpha}(-z),$$

we have

$$\begin{aligned}
\|S_\alpha(t)\| &\le \int_0^\infty \phi_\alpha(\theta)\|S(\theta t^\alpha)\| d\theta \\
&\le M\int_0^\infty \phi_\alpha(\theta) e^{-at^\alpha\theta} d\theta = ME_{\alpha,1}(-at^\alpha), \\
\|P_\alpha(t)\| &\le \int_0^\infty \alpha\theta\phi_\alpha(\theta)\|S(\theta t^\alpha)\| d\theta \\
&\le M\int_0^\infty \alpha\theta\phi_\alpha(\theta) e^{-at^\alpha\theta} d\theta = ME_{\alpha,\alpha}(-at^\alpha).
\end{aligned}$$

On the other hand, we have the following asymptotic expansion for $E_{\alpha,\beta}$ as $z \to \infty$ (see, e.g., [16]):

$$E_{\alpha,\beta}(z) = \begin{cases} \frac{1}{\alpha} z^{(1-\beta)/\alpha} \exp z^{1/\alpha} + \varepsilon_{\alpha,\beta}(z) & \text{if } |\arg z| \le \frac{1}{2}\pi\alpha, \\ \varepsilon_{\alpha,\beta}(z) & \text{if } |\arg(-z)| \le (1-\frac{1}{2}\alpha)\pi, \end{cases}$$

where

$$\varepsilon_{\alpha,\beta}(z) = -\sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(\beta-\alpha n)} + O(|z|^{-N}), \text{ as } z \to \infty.$$

Thus, in our case

$$\|S_\alpha(t)\| \le ME_{\alpha,1}(-at^\alpha) = M\varepsilon_{\alpha,1}(-at^\alpha),$$
$$\|P_\alpha(t)\| \le ME_{\alpha,\alpha}(-at^\alpha) = M\varepsilon_{\alpha,\alpha}(-at^\alpha).$$

Two last inequalities ensure that $\|S_\alpha(t)\|$,$\|P_\alpha(t)\| = O(t^{-\alpha})$ as $t \to \infty$. The proposition is proved. □

Consider the linear Cauchy problem

$$^C D_0^\alpha x(t) = Ax(t) + f(t), t \in J,$$
$$x(0) = x_0.$$

By using the Laplace transform for fractional derivative (see [20, Lemma 2.24]), we have

$$(\lambda^\alpha - A)\widehat{x}(\lambda) = \lambda^{\alpha-1}x_0 + \widehat{f}(\lambda),$$

where $\widehat{y}$ stands for the Laplace transform of $y \in L^1(0,\infty;\mathbb{R}^n)$. Then for $\lambda \in \mathbb{R}^+$ such that $\lambda^\alpha \in \rho(A)$ (the resolvent set of A), we have

$$\widehat{x}(\lambda) = \lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x_0 + (\lambda^\alpha - A)^{-1}\widehat{f}(\lambda).$$

Taking the inverse Laplace transform of the last equation, one has the following representation of x:

$$x(t) = S_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1}P_\alpha(t-s)f(s)ds, \tag{2.11}$$

thanks to the fact that

$$\begin{aligned}\widehat{S_\alpha}(\lambda) &= \lambda^{\alpha-1}(\lambda^\alpha - A)^{-1},\\ \widehat{(\cdot)^{\alpha-1}P_\alpha}(\lambda) &= (\lambda^\alpha - A)^{-1}.\end{aligned}$$

3. Existence result on compact intervals

Denote

$$\begin{aligned}C_T &= C([0,T];\mathbb{R}^n),\\ C_\tau &= C([-\tau,0];\mathbb{R}^n),\\ \mathcal{C} &= C([-\tau,T];\mathbb{R}^n).\end{aligned}$$

Let χ_T and χ_τ be the Hausdorff MNCs on $\mathcal{C}$ and C_τ, respectively. In the formulation of problem (1.1)-(1.3), we assume that

(H1) *A is a linear operator on $\mathbb{R}^n$.*

(H2) *$B : [0,T]\times\mathbb{R}^n\times C_\tau \to \mathbb{R}^{n\times m}$, $F : [0,T]\times\mathbb{R}^n \to \mathbb{R}^m$, $G : K \to \mathbb{R}^m$, and $h : \mathcal{C} \to C_\tau$ are continuous maps such that*

(1) *there exist $\eta_B \in L^p(0,T), p > \frac{1}{\alpha}$, and a non-decreasing continuous function Ψ_B such that*

$$\|B(t,v,w)\| \le \eta_B(t)\Psi_B(\|v\| + \|w\|_{C_\tau})$$

for all $v \in \mathbb{R}^n, w \in C_\tau$;

(2) *there is a positive number η_F such that $\|F(t,v)\| \le \eta_F$ for all $t \in [0,T]$ and $v \in \mathbb{R}^n$;*

(3) *G is monotone on K, that is*

$$\langle u-v, G(u)-G(v)\rangle \ge 0,\ \forall u,v \in K;$$

(4) *there exists $v_0 \in K$ such that*

$$\lim_{v\in K,\|v\|\to\infty} \frac{\langle v-v_0, G(v)\rangle}{\|v\|^2} > 0;$$

(5) *there is a non-decreasing continuous function Ψ_h such that*

$$\|h(x)\|_{C_\tau} \le \Psi_h(\|x\|_{\mathcal{C}}),\ \forall x \in \mathcal{C};$$

(6) *there exists* $\eta_h \geq 0$ *such that*

$$\chi_\tau(h(\Omega)) \leq \eta_h \chi_T(\Omega),$$

for all bounded sets $\Omega \subset \mathcal{C}$.

Remark 3.1. Let us give some comments on assumption (H2)(6). As mentioned in [18], this condition is satisfied if h is a Lipschitz function. Precisely, if

$$\|h(x_1) - h(x_2)\|_{C_\tau} \leq \eta_h \|x_1 - x_2\|_{\mathcal{C}},$$

then (H2)(6) is true. On the other hand, in the non-delay case ($\tau = 0$), we can remove this assumption. Indeed, if $\tau = 0$ then $h(\Omega)$ is a bounded set in $\mathbb{R}^n$ thanks to (H2)(5) and so it is relatively compact. Then (H2)(6) is fulfilled with $\eta_h = 0$.

Motivated by formula (2.11), we have the following definition.

Definition 3.1. By a mild solution to (1.1)-(1.3) on $[-\tau, T]$, we mean a function $x \in \mathcal{C}$, for which there exists an integrable function $u : J \to K$ such that

$$x(t) = S_\alpha(t)[\varphi(0) - h(x)(0)] + \int_0^t (t-s)^{\alpha-1} P_\alpha(t-s) B(s, x(s), x_s) u(s) ds, \quad t \in J,$$

$$\langle v - u(t), F(t, x(t)) + G(u(t)) \rangle \geq 0, \quad \text{for a.e. } t \in J, \ \forall v \in K,$$

$$x(s) + h(x)(s) = \varphi(s), \quad s \in [-\tau, 0].$$

Now for a given mapping $Q : \mathbb{R}^m \to \mathbb{R}^m$, we denote

$$SOL(K, Q) = \{v \in K : \langle w - v, Q(v) \rangle \geq 0, \forall w \in K\}. \tag{3.1}$$

Due to [25, Proposition 6.2], one has the following result.

Lemma 3.1 ([25]). *Let (H2)(3)-(H2)(4) hold. Then for every* $z \in \mathbb{R}^m$, *the solution set* $SOL(K, z + G(\cdot))$ *is non-empty, convex and closed. Moreover, there exists* $\eta_G > 0$ *such that*

$$\|v\| \leq \eta_G(1 + \|z\|), \ \forall v \in SOL(K, z + G(\cdot)). \tag{3.2}$$

In order to solve (1.1)-(1.3), we convert it to a differential inclusion. Let

$$U(z) = SOL(K, z + G(\cdot)), \ z \in \mathbb{R}^m.$$

Then it is easy to verify that $U:\mathbb{R}^m \to \mathcal{P}(\mathbb{R}^m)$ has closed convex values. In addition, U is a closed multimap. From (3.2) we see that U is locally bounded, then it is u.s.c.

Now we define $\Phi : J\times\mathbb{R}^n\times C_\tau \to \mathcal{P}(\mathbb{R}^n)$ as follows:

$$\Phi(t,v,w) = \{B(t,v,w)y : y\in U(F(t,v))\}. \tag{3.3}$$

Since U has closed convex values, so does Φ. Furthermore, thanks to the continuity of B and F, the composition multimap Φ is u.s.c. For $x\in\mathcal{C}$, we denote

$$\mathcal{P}_\Phi(x) = \{f\in L^p(J;\mathbb{R}^n) : f(t)\in\Phi(t,x(t),x_t) \text{ for a.e. } t\in J\}.$$

It turns out that the solution of (1.1)-(1.3) is given by

$$x(t) = S_\alpha(t)[\varphi(0)-h(x)(0)] + \int_0^t (t-s)^{\alpha-1}P_\alpha(t-s)f(s)ds,\ f\in\mathcal{P}_\Phi(x),\ t\in J, \tag{3.4}$$

$$x(t)+h(x)(t) = \varphi(t),\ t\in[-\tau,0]. \tag{3.5}$$

Let $W : L^p(J;\mathbb{R}^n)\to C_T$ be the operator defined as

$$W(f)(t) = \int_0^t (t-s)^{\alpha-1}P_\alpha(t-s)f(s)ds. \tag{3.6}$$

We are now in a position to define *the solution multioperator* $\Sigma:\mathcal{C}\to\mathcal{P}(\mathcal{C})$ as follows: for given $\varphi\in C_\tau$,

$$\Sigma(x)(t) = \begin{cases} \varphi(t)-h(x)(t), & t\in[-\tau,0],\\ \{S_\alpha(t)[\varphi(0)-h(x)(0)] + W(f)(t) : f\in\mathcal{P}_\Phi(x)\}, & t\in J. \end{cases} \tag{3.7}$$

Then $x\in\mathcal{C}$ is a solution of (3.4)-(3.5) iff x is a fixed point of Σ. We will apply Theorem 2.1 to show that $Fix(\Sigma)\neq\emptyset$. At first, we have to prove some necessary properties of the solution multioperator.

Using Lemma 2.3, we show that $\mathcal{P}_\Phi$ is weakly u.s.c.

LEMMA 3.2. *Under the assumptions (H2)(1)-(H2)(4), $\mathcal{P}_\Phi$ is well-defined and weakly u.s.c.*

P r o o f. Using the assumptions and the result of Lemma 3.1, we get

$$\begin{aligned}\|\Phi(t,v,w)\| &:= \sup\{\|z\| : z\in\Phi(t,v,w)\}\\ &\le \|B(t,v,w)\|\,\eta_G(1+\|F(t,v)\|)\\ &\le \eta_G(1+\eta_F)\eta_B(t)\,\Psi_B(\|v\|+\|w\|_{C_\tau}).\end{aligned} \tag{3.8}$$

Since Φ is u.s.c with compact convex values, the multimap $\Lambda(t) = \Phi(t, x(t), x_t)$ is (strongly) measurable due to [18, Proposition 1.3.1]. Thus by [18, Theorem 1.3.1], it has the Castaing representation (see [18, Definition 1.3.3]) and hence $\mathcal{P}_\Phi(x) \neq \emptyset$ for $x \in \mathcal{C}$.

To prove the second assertion, we use Lemma 2.3. Let $\{x_k\} \subset \mathcal{C}$ be such that $x_k \to x^*$, $f_k \in \mathcal{P}_\Phi(x_k)$. We see that $\{f_k(t)\} \subset C(t) := \Phi(t, \overline{\{x_k(t), (x_k)_t\}})$, and $C(t)$ is a compact set for each $t \in J$. Furthermore, by (3.8) the sequence $\{f_k\}$ is integrably bounded (i.e., bounded by an L^p-integrable function). Therefore $\{f_k\}$ is weakly relatively compact in $L^p(J; \mathbb{R}^n)$ (see [10]). Let $f_k \rightharpoonup f^*$. Then by Mazur's lemma (see, e.g. [11]), there are $\tilde{f}_k \in \operatorname{co}\{f_i : i \geq k\}$ such that $\tilde{f}_k \to f^*$ in $L^p(J; \mathbb{R}^n)$ and then $\tilde{f}_k(t) \to f^*(t)$ for a.e. $t \in J$, up to a subsequence. Observe that in our case, the upper semicontinuity of Φ yields that

$$\Phi(t, x_k(t), (x_k)_t) \subset \Phi(t, x^*(t), x_t^*) + B_\epsilon,$$

for all sufficiently large k, here $\epsilon > 0$ is given and B_ϵ is the ball in $\mathbb{R}^n$ centered at origin with radius ϵ. So

$$f_k(t) \in \Phi(t, x^*(t), x_t^*) + B_\epsilon, \text{ for a.e. } t \in J,$$

and the same inclusion holds for $\tilde{f}_k(t)$ thanks to the convexity of $\Phi(t, x^*(t), x_t^*) + B_\epsilon$. Hence, $f^*(t) \in \Phi(t, x^*(t), x_t^*) + B_\epsilon$ for a.e. $t \in J$. Since ϵ is arbitrary, one gets $f^* \in \mathcal{P}_\Phi(x^*)$. The lemma is proved. □

Proposition 3.1 ([27, Lemma 1]). *Let $P(t,s)$ be a family of linear operators on $\mathbb{R}^n$ for $t, s \in J, s \leq t$. Assume that P satisfies the following conditions:*

- (P1) *there exists a function $\rho \in L^q(J; \mathbb{R}), q > 1$ such that $\|P(t,s)\| \leq \rho(t-s)$ for all $t, s \in J, s \leq t$;*
- (P2) *$\|P(t,s) - P(r,s)\| \leq \epsilon$ for $0 \leq s \leq r - \epsilon, r < t = r + h \leq T$ with $\epsilon = \epsilon(h) \to 0$ as $h \to 0$.*

Then the operator $\mathbf{S} : L^{q'}(J; \mathbb{R}^n) \to C(J; \mathbb{R}^n)$ defined by

$$(\mathbf{S}g)(t) := \int_0^t P(t,s) g(s) ds$$

maps any bounded set to an equicontinuous one, where q' is the conjugate of $q : \frac{1}{q} + \frac{1}{q'} = 1$.

Lemma 3.3. *The operator W defined by (3.6) is compact.*

P r o o f. We show that $W(\Omega)$ is relatively compact in C_T for any bounded set $\Omega \subset L^p(J;\mathbb{R}^n)$. Obviously, $W(\Omega)(t)$ is bounded for each $t \in J$. Moreover, the operator

$$Q(t,s) = (t-s)^{\alpha-1}P_\alpha(t-s)$$

satisfies the assumption of Proposition 3.1. Then it follows that $W(\Omega)$ is an equicontinuous set. Therefore the conclusion follows by the Arzela-Ascoli theorem. □

Now we can show some properties of the solution multioperator Σ.

LEMMA 3.4. *Let (H1), (H2)(1)-(H2)(4) hold. Then the solution multioperator Σ is quasicompact and closed.*

P r o o f. Since h is continuous and W is compact, it is easy to check that $\Sigma(K)$ is relatively compact for any compact set $K \subset \mathcal{C}$. So it is a quasicompact multimap.

Now let $\{x_k\} \subset \mathcal{C}, x_k \to x^*, y_k \in \Sigma(x_k)$ and $y_k \to y^*$. We will verify that $y^* \in \Sigma(x^*)$. Take $f_k \in \mathcal{P}_\Phi(x_k)$ such that

$$y_k(t) = \varphi(t) - h(x_k)(t), t \in [-\tau, 0], \tag{3.9}$$

$$y_k(t) = S_\alpha(t)[\varphi(0) - h(x_k)(0)] + W(f_k)(t), t \in J. \tag{3.10}$$

Since $\mathcal{P}_\Phi$ is weakly u.s.c and $\{x_k\}$ is compact, $\{f_k\}$ is weakly compact and one can assume that $f_k \rightharpoonup f^*$ in $L^p(J;\mathbb{R}^n)$. Moreover, $f^* \in \mathcal{P}_\Phi(x^*)$. By the compactness of W, we obtain $W(f_k) \to W(f^*)$ in C_T. Taking the limits of (3.9)-(3.10) as $k \to \infty$, we get

$$y^*(t) = \varphi(t) - h(x^*)(t), t \in [-\tau, 0],$$
$$y^*(t) = S_\alpha(t)[\varphi(0) - h(x^*)(0)] + W(f^*)(t), t \in J, f^* \in \mathcal{P}_\Phi(x^*).$$

Thus $y^* \in \Sigma(x^*)$. The proof is complete. □

LEMMA 3.5. *Assume (H1)-(H2). If $\eta_h S_\alpha^T < 1$ then Σ is χ_T-condensing, here $S_\alpha^T = \sup\limits_{t\in J} \|S_\alpha(t)\|$.*

P r o o f. Let $D \subset \mathcal{C}$ be a bounded set. Then we have

$$\Sigma(D) = \Sigma_1(D) + \Sigma_2(D),$$

where

$$\Sigma_1(x)(t) = \begin{cases} S_\alpha(t)[\varphi(0) - h(x)(0)], & t \in J, \\ \varphi(t) - h(x)(t), & t \in [-\tau, 0]; \end{cases}$$

$$\Sigma_2(x)(t) = \begin{cases} \{W(f)(t) : f \in \mathcal{P}_\Phi(x)\}, & t \in J, \\ 0, & t \in [-\tau, 0]. \end{cases}$$

By the algebraic semi-additivity property of χ_T, we have

$$\chi_T(\Sigma(D)) \leq \chi_T(\Sigma_1(D)) + \chi_T(\Sigma_2(D)). \tag{3.11}$$

For $z_1, z_2 \in \Sigma_1(D)$, there exist $x_1, x_2 \in D$ such that

$$z_1(t) = \begin{cases} S_\alpha(t)[\varphi(0) - h(x_1)(0)], & t \in J, \\ \varphi(t) - h(x_1)(t), & t \in [-\tau, 0], \end{cases}$$

$$z_2(t) = \begin{cases} S_\alpha(t)[\varphi(0) - h(x_2)(0)], & t \in J, \\ \varphi(t) - h(x_2)(t), & t \in [-\tau, 0]. \end{cases}$$

Then

$$\|z_1(t) - z_2(t)\| \leq \begin{cases} \|S_\alpha(t)\| \|h(x_1) - h(x_2)\|_{C_\tau}, & t \in J, \\ \|h(x_1) - h(x_2)\|_{C_\tau}, & t \in [-\tau, 0]. \end{cases}$$

Therefore

$$\|z_1 - z_2\|_{\mathcal{C}} \leq S_\alpha^T \|h(x_1) - h(x_2)\|_{C_\tau},$$

thanks to the fact that $S_\alpha^T \geq 1$. This implies

$$\chi_T(\mathcal{F}_1(D)) \leq S_\alpha^T \chi_\tau(h(D)).$$

Employing (H2)(6), we have

$$\chi_T(\mathcal{F}_1(D)) \leq \eta_h S_\alpha^T \, \chi_T(D). \tag{3.12}$$

Concerning Σ_2, we first observe that $\mathcal{P}_\Phi(D)$ is bounded due to estimate (3.8). Then using the compactness of W, we see that $\Sigma_2(D)$ is a relatively compact set. Then $\chi_T(\Sigma_2(D)) = 0$. Combining (3.11)-(3.12), we have

$$\chi_T(\Sigma(D)) \leq \eta_h S_\alpha^T \, \chi_T(D).$$

Now if $\chi_T(D) \leq \chi_T(\Sigma(D))$ then $\chi_T(D) \leq \eta_h S_\alpha^T \, \chi_T(D)$. This implies $\chi_T(D) = 0$, thanks to the assumption that $\eta_h S_\alpha^T < 1$. By the regularity of χ_T, we have that D is relatively compact. The proof is complete. □

Now we can state the main result of this section.

THEOREM 3.1. *Assume* $(H1)-(H2)$. *Then problem* (3.4)-(3.5) *has at least one mild solution on* $[-\tau, T]$ *provided* $\eta_h S_\alpha^T < 1$ *and*

$$\liminf_{r\to\infty}\Big[S_\alpha^T\frac{\Psi_h(r)}{r}+\eta_G(1+\eta_F)\frac{\Psi_B(2r)}{r}\times\sup_{t\in J}\int_0^t(t-s)^{\alpha-1}\|P_\alpha(t-s)\|\eta_B(s)ds\Big]<1. \quad (3.13)$$

P r o o f. The assumption $\eta_h S_\alpha^T < 1$ ensures that Σ is χ_T-condensing. In addition, by Lemma 3.4 and Lemma 2.2, Σ is u.s.c. In order to apply Theorem 2.1, it suffices to show that there exists $R > 0$ such that $\Sigma(B_R) \subset B_R$, where B_R is the ball in $\mathcal{C}$ centered at origin with radius R.

Assume to the contrary that there exists a sequence $\{x_k\} \subset \mathcal{C}$ such that $\|x_k\|_{\mathcal{C}} \le k$ and $y_k \in \Sigma(x_k)$ such that $\|y_k\|_{\mathcal{C}} > k$. By the definition of Σ, one can take $f_k \in \mathcal{P}_\Phi(x_k)$ such that

$$y_k(t) = \varphi(t) - h(x_k)(t), t \in [-\tau, 0],$$
$$y_k(t) = S_\alpha(t)[\varphi(0) - h(x_k)(0)] + W(f_k)(t), t \in J.$$

Then for $t \in [-\tau, 0]$ we have

$$\begin{aligned}\|y_k(t)\| &\le \|\varphi\|_{C_\tau} + \|h(x_k)\|_{C_\tau}\\ &\le \|\varphi\|_{C_\tau} + \Psi_h(\|x_k\|_{\mathcal{C}}) \le \|\varphi\|_{C_\tau} + \Psi_h(k)\end{aligned}$$

thanks to (H2)(5). For $t \in J$ we get

$$\begin{aligned}\|y_k(t)\| &\le S_\alpha^T(\|\varphi\|_{C_\tau} + \|h(x_k)\|_{C_\tau}) + \sup_{t\in J}\|W(f_k)(t)\|\\ &\le S_\alpha^T[\|\varphi\|_{C_\tau} + \Psi_h(\|x_k\|_{\mathcal{C}})] + \sup_{t\in J}\int_0^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\|\,\|f_k(s)\|ds\\ &\le S_\alpha^T[\|\varphi\|_{C_\tau} + \Psi_h(k)]\\ &\qquad + \eta_G(1+\eta_F)\Psi_B(2k)\sup_{t\in J}\int_0^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\|\eta_B(s)ds,\end{aligned}$$

thanks to estimate (3.8). Since $S_\alpha^T \ge 1$, we obtain

$$\|y_k\|_{\mathcal{C}} \le S_\alpha^T[\|\varphi\|_{C_\tau}+\Psi_h(k)]+\eta_G(1+\eta_F)\Psi_B(2k)\sup_{t\in J}\int_0^t(t-s)^{\alpha-1}\|P_\alpha(t-s)\|\eta_B(s)ds.$$

Then

$$\begin{aligned}&\liminf_{k\to\infty}\frac{\|y_k\|_{\mathcal{C}}}{k}\\ &\le \liminf_{k\to\infty}[S_\alpha^T\frac{\Psi_h(k)}{k}+\eta_G(1+\eta_F)\frac{\Psi_B(2k)}{k}\sup_{t\in J}\int_0^t(t-s)^{\alpha-1}\|P_\alpha(t-s)\|\eta_B(s)ds].\end{aligned}$$

Thus $\liminf_{k\to\infty} \frac{\|y_k\|_{\mathcal{C}}}{k} < 1$ due to assumption (3.13), and we get a contradiction. The proof is complete. □

4. **Decay solutions**

For a positive number γ, denote

$$BC_\gamma = \{x \in C([-\tau,\infty);\mathbb{R}^n) : t^\gamma\|x(t)\| = O(1) \text{ as } 0 < t \to \infty\}.$$

Then BC_γ is a subspace of BC_0, here $BC_0 := BC_0([-\tau,\infty);\mathbb{R}^n)$ is the space of continuous functions on $[-\tau,\infty)$ vanishing at infinity. It should be noted that BC_0 with the sup norm

$$\|x\|_\infty = \sup_{t\geq -\tau} \|x(t)\|,$$

is a Banach space. In this section, we prove the existence of solutions to (3.4)-(3.5) in BC_γ. We need the following additional assumption:

(H3) *there exists $a > 0$ such that $\langle -Az, z\rangle \geq a\|z\|^2$ for all $z \in \mathbb{R}^n$.*

In addition, we replace (H2) with a stronger one:

(H2') *The functions B, F, G and h satisfy (H2) for all $T > 0$, with $\Psi_B(r) = r$ and $\Psi_h(r) = \nu_h r$, where ν_h is a positive number.*

We first have the following lemma.

Lemma 4.1. *Let the hypotheses (H1), (H2') and (H3) hold. Then there exists $\rho > 0$ such that $\Sigma(B_\rho) \subset B_\rho$ provided that*

$$\nu_h S_\alpha^\infty + 2\eta_G(1+\eta_F)\sup_{t\geq 0}\int_0^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\|\eta_B(s)ds < 1, \tag{4.1}$$

where B_ρ is the closed ball in $BC_0([-\tau,\infty);\mathbb{R}^n)$ with center at origin and radius ρ, and $S_\alpha^\infty = \sup_{t\geq 0}\|S_\alpha(t)\|$.

P r o o f. The proof is similar to those as in the proof of Theorem 3.1, but now with $\Psi_B(r) = r$ and $\Psi_h(r) = \nu_h r$. □

Taking ρ from Lemma 4.1, we consider the following bounded closed and convex set in $BC_0([-\tau,\infty);\mathbb{R}^n)$:

$$B_\rho^\alpha(R) = B_\rho \cap \{x \in BC_\alpha : t^\alpha\|x(t)\| \leq R,\ \forall t \geq 0\}. \tag{4.2}$$

Lemma 4.2. *Let the hypotheses (H1), (H2') and (H3) hold. Then there exists $R > 0$ such that*

$$\Sigma(B_\rho^\alpha(R)) \subset B_\rho^\alpha(R),$$

provided that (4.1) *is satisfied and*

$$\sup_{t\geq 0}\int_0^{\frac{t}{2}}(t-s)^{\alpha-1}\eta_B(s)ds<\infty, \tag{4.3}$$

$$2^{\alpha+1}\eta_G(1+\eta_F)\sup_{t\geq 0}\int_{\frac{t}{2}}^t(t-s)^{\alpha-1}\|P_\alpha(t-s)\|\eta_B(s)ds<1. \tag{4.4}$$

P r o o f. By (H3), we have

$$\|e^{tA}\|\leq e^{-at}, t\geq 0.$$

So it follows from Proposition 2.1 that $\|S_\alpha(t)\|, \|P_\alpha(t)\| = O(t^{-\alpha})$ as $t\to\infty$. Assume to the contrary that for each $n=1,2,...$ there exist $x_n\in B^\alpha_\rho(n)$ and $y_n\in\Sigma(x_n)$ with

$$\sup_{t\geq 0}t^\alpha\|y_n(t)\|>n. \tag{4.5}$$

Then one can find $f_n\in\mathcal{P}_\Phi(x_n)$ such that

$$y_n(t)=S_\alpha(t)[\varphi(0)-h(x_n)(0)]+\int_0^t(t-s)^{\alpha-1}P_\alpha(t-s)f_n(s)ds, \forall t>0.$$

Using (H2') and estimate (3.8), we have

$$\begin{aligned}\|y_n(t)\|\leq{}&\|S_\alpha(t)\|(\|\varphi\|_{C_\tau}+\nu_h\|x_n\|_\infty)\\&+\eta_G(1+\eta_F)\int_0^t(t-s)^{\alpha-1}\|P_\alpha(t-s)\|\eta_B(s)(\|x_n(s)\|+\|(x_n)_s\|_{C_\tau})ds\\={}&I_1(t)+\eta_G(1+\eta_F)[I_2(t)+I_3(t)],\end{aligned} \tag{4.6}$$

where

$$I_1(t)=\|S_\alpha(t)\|(\|\varphi\|_{C_\tau}+\nu_h\|x_n\|_\infty),$$

$$I_2(t)=\int_0^{\frac{t}{2}}(t-s)^{\alpha-1}\|P_\alpha(t-s)\|\eta_B(s)(\|x_n(s)\|+\|(x_n)_s\|_{C_\tau})ds,$$

$$I_3(t)=\int_{\frac{t}{2}}^t(t-s)^{\alpha-1}\|P_\alpha(t-s)\|\eta_B(s)(\|x_n(s)\|+\|(x_n)_s\|_{C_\tau})ds.$$

Since $\|S_\alpha(t)\|=O(t^{-\alpha})$ as $t\to\infty$, we have

$$\sup_{t\geq 0}t^\alpha I_1(t)\leq S^*_\alpha(\|\varphi\|_{C_\tau}+\nu_h\rho), \tag{4.7}$$

where $S^*_\alpha=\sup_{t\geq 0}t^\alpha\|S_\alpha(t)\|$. Considering I_2, we have

$$\begin{aligned}I_2(t)&=\int_0^{\frac{t}{2}}(t-s)^{\alpha-1}\|P_\alpha(t-s)\|\eta_B(s)(\|x_n(s)\|+\|(x_n)_s\|_{C_\tau})ds\\&\leq 2\rho\int_0^{\frac{t}{2}}(t-s)^{\alpha-1}\|P_\alpha(t-s)\|\eta_B(s)ds.\end{aligned} \tag{4.8}$$

For $s \le \frac{t}{2}$, one has

$$\|P_\alpha(t-s)\| \le C(t-s)^{-\alpha} \le C\left(\frac{t}{2}\right)^{-\alpha}, \text{ for some } C > 0. \tag{4.9}$$

Hence it follows from (4.8)-(4.9) that

$$\sup_{t\ge 0} t^\alpha I_2(t) \le 2^{\alpha+1}\rho C \sup_{t\ge 0}\int_0^{\frac{t}{2}} (t-s)^{\alpha-1}\eta_B(s)ds, \tag{4.10}$$

where the last term is finite due to (4.3). Now one observes that $t^\alpha\|x_n(t)\| \le n$ for any $t \ge 0$. Then for any $\sigma > 0$ and $t > \tau+\sigma$, we get

$$\begin{aligned} t^\alpha\|(x_n)_t\|_{C_\tau} &= t^\alpha \sup_{\zeta\in[-\tau,0]} \|x_n(t+\zeta)\| \\ &= t^\alpha \sup_{\zeta\in[-\tau,0]} (t+\zeta)^{-\alpha}(t+\zeta)^\alpha\|x_n(t+\zeta)\| \\ &\le t^\alpha(t-\tau)^{-\alpha} \sup_{\zeta\in[-\tau,0]} (t+\zeta)^\alpha\|x_n(t+\zeta)\| \\ &\le n\left(\frac{\tau+\sigma}{\sigma}\right)^\alpha. \end{aligned}$$

For $t \in [0, \tau+\sigma]$ we have $t^\alpha\|(x_n)_t\|_{C_\tau} \le (\tau+\sigma)^\alpha\rho$. Then

$$t^\alpha\|(x_n)_t\|_{C_\tau} \le n\left(\frac{\tau+\sigma}{\sigma}\right)^\alpha + (\tau+\sigma)^\alpha\rho,\ \forall t \ge 0.$$

The last inequality allows us to estimate I_3 as follows

$$\begin{aligned} t^\alpha I_3(t) &= t^\alpha\int_{\frac{t}{2}}^t s^{-\alpha}(t-s)^{\alpha-1}\|P_\alpha(t-s)\|\eta_B(s)(s^\alpha\|x_n(s)\|+s^\alpha\|(x_n)_s\|_{C_\tau})ds \\ &\le 2^\alpha z_n(\sigma)\int_{\frac{t}{2}}^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\|\eta_B(s)ds, \end{aligned} \tag{4.11}$$

where

$$z_n(\sigma) = n + n\left(\frac{\tau+\sigma}{\sigma}\right)^\alpha + (\tau+\sigma)^\alpha\rho.$$

Taking (4.6) into account and using (4.7), (4.10) and (4.11), we arrive at

$$\lim_{n\to\infty}\frac{1}{n}\sup_{t\geq 0} t^\alpha\|y_n(t)\| \leq \lim_{n\to\infty}\frac{1}{n}\left[\sup_{t\geq 0} t^\alpha I_1(t) + \eta_G(1+\eta_F)\sup_{t\geq 0} t^\alpha I_2(t)\right]$$
$$+ \lim_{n\to\infty}\frac{1}{n}\eta_G(1+\eta_F)\sup_{t\geq 0} t^\alpha I_3(t)$$
$$= \eta_G(1+\eta_F)\lim_{n\to\infty}\frac{1}{n}\sup_{t\geq 0} t^\alpha I_3(t)$$
$$\leq 2^\alpha\left[1+\left(\frac{\tau+\sigma}{\sigma}\right)^\alpha\right]\eta_G(1+\eta_F)\sup_{t\geq 0}\int_{\frac{t}{2}}^t (t-s)^{\alpha-1}\|P_\alpha(t-s)\|\eta_B(s)ds.$$

By (4.4), one can choose $\sigma > 0$ such that $\lim_{n\to\infty}\frac{1}{n}\sup_{t\geq 0} t^\alpha\|y_n(t)\| < 1$ which contradicts (4.5). The proof is complete. □

By the last lemma, from now on one can consider

$$\Sigma : B^\alpha_\rho(R) \to \mathcal{P}(B^\alpha_\rho(R)).$$

We need the condensivity property of Σ.

Lemma 4.3. *Let the hypotheses of Lemma 4.2 hold. Then Σ is χ^*-condensing provided $\eta_h S^\infty_\alpha < 1$, where $S^\infty_\alpha = \sup_{t\geq 0}\|S_\alpha(t)\|$.*

P r o o f. Let $D \subset B^\alpha_\rho(R)$ be a bounded set. Recall that $\chi^*(D) = \chi_\infty(D) + d_\infty(D)$, where χ_∞ and d_∞ are defined by (2.2) and (2.4), respectively. By the same arguments as in the proof of Lemma 3.5, we have

$$\chi_T(\pi_T(\Sigma(D))) \leq \eta_h \sup_{t\in[0,T]}\|S_\alpha(t)\|\chi_T(\pi_T(D)), \forall T > 0.$$

Therefore

$$\chi_\infty(\Sigma(D)) \leq \eta_h S^\infty_\alpha \chi_\infty(D). \tag{4.12}$$

In the rest of the proof, we will show that $d_\infty(\Sigma(D)) = 0$. Obviously, for any $y \in \Sigma(D)$ we have $\|y(t)\| \leq Rt^{-\alpha}$ for all $t \geq T > 0$. So

$$\sup_{t\geq T}\|y(t)\| \leq RT^{-\alpha}.$$

This implies $d_T(D) \leq RT^{-\alpha}$, and hence $d_\infty(D) = \lim_{T\to\infty} d_T(D) = 0$. This assertion together with (4.12) yields

$$\chi^*(\Sigma(D)) \leq \eta_h S^\infty_\alpha \chi^*(D).$$

Since χ^* is regular due to Lemma 2.1, the last inequality guarantees that Σ is χ^*-condensing as desired. □

LEMMA 4.4. *Let the hypotheses of Lemma 4.2 hold. Then* $\Sigma : B_\rho^\alpha(R) \to \mathcal{P}(B_\rho^\alpha(R))$ *is u.s.c.*

P r o o f. We apply Lemma 2.2 again. We first show the closedness of Σ. Let $x_k \in B_\rho^\alpha(R), x_k \to x^*$, $y_k \in \Sigma(x_k)$ and $y_k \to y^*$ in $B_\rho^\alpha(R)$. We prove that $y^* \in \Sigma(x^*)$, i.e.

$$y^*(t) \in \Sigma(x^*)(t), \forall t \in \mathbb{R}^+.$$

Let $t > 0$, then taking $T > t$ and arguing as in the proof of Lemma 3.4, we get that Σ is closed.

It remains to check the quasi-compactness of Σ. Let $K \subset B_\rho^\alpha(R)$ be a compact set and $\{y_k\} \subset \Sigma(K)$. Then there exists $\{x_k\} \subset K$ such that $y_k \in \Sigma(x_k)$. Let $f_k \in \mathcal{P}_\Phi(x_k)$ be such that

$$y_k(t) = \varphi(t) - h(x_k)(t),\ t \in [-\tau, 0],$$
$$y_k(t) = S_\alpha(t)x_k(0) + W(f_k)(t),\ t > 0.$$

Obviously, for all $T > 0$ we have that $\{\pi_T(y_k)\}$ is relatively compact. Therefore

$$\chi_\infty(\{y_k\}) = \sup_{T>0} \chi_T(\{\pi_T(y_k)\}) = 0. \tag{4.13}$$

Since $y_k \in B_\rho^\alpha(R)$, by using the same arguments as in the proof of Lemma 4.3 we obtain

$$d_\infty(\{y_k\}) = 0. \tag{4.14}$$

It follows from (4.13)-(4.14) that $\chi^*(\{y_k\}) = 0$. Then $\{y_k\}$ is relatively compact, thanks to Lemma 2.1.

Finally, the closedness and quasi-compactness of Σ imply that this operator is u.s.c on $B_\rho^\alpha(R)$. □

The following theorem is our main result.

THEOREM 4.1. *Let the hypotheses of Lemma 4.2 hold. If* $\eta_h S_\alpha^\infty < 1$ *then the problem* (3.4)-(3.5) *has a compact set of solutions on* $[-\tau, \infty)$ *satisfying*

$$t^\alpha \|x(t)\| = O(1) \text{ as } t \to \infty.$$

P r o o f. If $\eta_h S_\alpha^\infty < 1$ then Σ is χ^*-condensing due to Lemma 4.3. By Lemma 4.4, Σ is u.s.c. Employing Theorem 2.1 again, we get the conclusion. □

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References

[1] J.-P. Aubin, A. Cellina, *Differential Inclusions. Set-Valued Maps and Viability Theory.* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], **264**, Springer-Verlag, Berlin (1984).

[2] R.R. Akhmerov, M.I. Kamenskii, A.S. Potapov, A.E. Rodkina, B.N. Sadovskii, *Measures of Noncompactness and Condensing Operators.* Birkhäuser, Boston-Basel-Berlin (1992).

[3] R.P. Agarwal, M. Benchohra, and S. Hamani, A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions. *Acta Appl. Math.* **109**, No 3 (2010), 973–1033.

[4] E.P. Avgerinos, N.S. Papageorgiou, Differential variational inequalities in $\mathbf{R}^N$. *Indian J. Pure Appl. Math.* **28**, No 9 (1997), 1267–1287.

[5] J. Banas, L. Olszowy, On a class of measures of noncompactness in banach algebras and their application to nonlinear integral equations. *J. Anal. Appl.* **28** (2009), 475–498.

[6] K. Balachandran, Yong Zhou, J. Kokila, Relative controllability of fractional dynamical systems with distributed delays in control. *Comput. Math. Appl.* **64** (2012), 3201–3209.

[7] K.Balachandran, J.Y. Park, J.J.Trujillo, Controllability of nonlinear fractional dynamical systems. *Nonlinear Anal.* **75** (2012), 1919–1926.

[8] K. Balachandran, V. Govindaraj, M. Rivero, J.A. Tenreiro Machado, J.J. Trujillo, Observability of nonlinear fractional dynamical systems. *Abstr. Appl. Anal.* (2013), Art. ID 346041.

[9] D. Bothe, Multivalued perturbations of m-accretive differential inclusions. *Israel J. Math.* **108** (1998), 109–138.

[10] J. Diestel, W.M. Ruess, W. Schachermayer, Weak compactness in $L^l(\mu, X)$. *Proc. Amer. Math. Soc.* **118** (1993), 447–453.

[11] I. Ekeland, R. Temam, *Convex Analysis and Variational Problems.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia – PA (1999).

[12] K.-J. Engel, R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*. Graduate Texts in Mathematics, # **194**, Springer-Verlag, New York (2000).

[13] Tian Liang Guo, The necessary conditions of fractional optimal control in the sense of Caputo. *J. Optim. Theory Appl.* **156** (2013), 115–126.

[14] J. Gwinner, On differential variational inequalities and projected dynamical systems - equivalence and a stability result. *Discrete Contin. Dynam. Syst.* (*Dynamical Systems and Differential Equations. Proc. of the 6th AIMS Internat. Conference*), Suppl. (2007), 467–476.

[15] J. Gwinner, A note on linear differential variational inequalities in hilbert spaces. *Syst. Model. Optim.* **391** (2013), 85–91.

[16] H.J. Haubold, A.M. Mathai, R.K. Saxena, Mittag-Leffler functions and their applications. *J. Appl. Math.* **2011** (2011), Art ID 298628, 51 pages.

[17] J.K. Hale, S.M. Verduyn Lunel, *Theory of Functional Differential Equations*. Springer-Verlag, New York (1993).

[18] M. Kamenskii, V. Obukhovskii, P. Zecca, *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces*. Walter de Gruyter, Berlin - New York (2001).

[19] T.D. Ke, V. Obukhovskii, N.C. Wong, J.C. Yao, On a class of fractional order differential inclusions with infinite delays. *Appl. Anal.* **92** (2013), 115–137.

[20] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*. Elsevier, Amsterdam (2006).

[21] V. Kiryakova, *Generalized Fractional Calculus and Applications*. Pitman Res. Notes in Math. Ser. #**301**, Longman Sci. and Techn., Harlow & John Wiley, New York (1994).

[22] Z. Liu, N.V. Loi, V. Obukhovskii, Existence and global bifurcation of periodic solutions to a class of differential variational inequalities. *Int. J. Bifur. Chaos.* **23**, No 7 (2013), # 1350125.

[23] K.S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*. Wiley-Intersci. Publ., John Wiley & Sons, Inc., New York (1993).

[24] V. Obukhovskii and J.-C. Yao, Some existence results for fractional functional differential equations. *Fixed Point Theory* **11**, No 1 (2010), 85–96.

[25] J.-S. Pang, D.E. Steward, Differential variational inequalities. *Math. Program. Ser. A* **113** (2008), 345–424.

[26] I. Podlubny, *Fractional Differential Equations. An Introduction to Fractional Derivatives Fractional Differential Equations, to Methods of*

Their Solution and Some of Their Applications. Math. in Science and Engin. # **198**, Academic Press, San Diego – CA (1999).

[27] T.I. Seidman, Invariance of the reachable set under nonlinear perturbations. *SIAM J. Control Optim.* **25** (1987), 1173–1191.

[28] R.-N. Wang, D.-H. Chena, T.-J. Xiao, Abstract fractional Cauchy problems with almost sectorial operators. *J. Differential Equations* **252** (2012), 202–235.

[29] Y. Zhou, F. Jiao, Existence of mild solutions for fractional neutral evolution equations. *Comp. Math. Appl.* **59** (2010), 1063–1077.

[1] *Department of Mathematics*
Hanoi National University of Education
136 Xuan Thuy, Cau Giay, Hanoi, VIETNAM
e-mail: ketd@hnue.edu.vn

[2] *Faculty of Fundamental Science*
PetroVietNam University
Ba Ria, VIETNAM
e-mail: loinv@pvu.edu.vn

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[3] *Faculty of Physics and Mathematics*
Voronezh State Pedagogical University
394043 Voronezh, RUSSIA
e-mail: valerio-ob2000@mail.ru

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